THE ASTROPHYSICAL JOURNAL, 458:783–801, 1996 February 20

EQUILIBRIUM OF TWISTED HORIZONTAL MAGNETIC FLUX TUBES

T. EMONET AND F. MORENO-INSERTIS
Instituto de Astrofísica de Canarias, E-38200 La Laguna (Tenerife), Spain
Received 1995 April 26; accepted 1995 August 22

ABSTRACT

The equilibrium of non–force-free twisted horizontal magnetic flux tubes is studied including gravity and an arbitrary pressure perturbation on the tube boundary. To solve this free-boundary problem, we use general nonorthogonal flux coordinates and consider the two-dimensional case in which there is no variation of the physical quantities along the tube axis. For the applications in the convection zone and corona, we consider the case of weak external stratification by assuming that the radius of the tube is smaller than the external pressure scale height. This allows us to introduce a perturbation scheme which is much less restrictive than the customary slender flux-tube approximation. In particular, it has the advantage of not imposing any limitation on the strength of the azimuthal field as compared to the longitudinal field. Within this scheme, one retains to zero order all the functional degrees of freedom of a general axisymmetric magnetostatic equilibrium; the geometry of the perturbed azimuthal field lines is then obtained from the equilibrium equations as a consequence of the zero-order density (or rather buoyancy) distribution in the tube and of the circular wavenumber of the external pressure perturbation. We show that, as a result of the presence of gravity, the field lines are no longer concentric, although they continue being circular. The resulting changes in magnetic pressure and tension of the azimuthal field exactly counteract the differences in buoyancy in the tube cross section. On the other hand, external pressure fluctuations of circular wavenumber higher than one can only be countered by bending the azimuthal field lines. In general terms, the present scheme allows one to study in detail the mutual dependence of the (differential) buoyancy in the tube, the intensity and field line geometry of the azimuthal magnetic field, and the gas pressure and longitudinal magnetic field distributions.

The main application of the equations and results of this paper is to study the transverse structure of magnetic flux rings embedded in a stratified medium with a flow around the tube that causes pressure fluctuations on its surface. This includes tubes in the deep convection zone, e.g., in its subadiabatic lower part, or those kept in place by a meridional flow. It also applies to flux rings moving in a quasi-static regime in which the drag force of the relative motion with respect to the external medium exactly compensates the total buoyancy of the tube. In this way, this study can complement the numerical simulations of the rise of magnetized tubes and bubbles toward the surface.

Subject headings: hydrodynamics — MHD — Sun: magnetic fields

1. INTRODUCTION

The problem of finding exact magnetohydrostatic configurations for the magnetic field and plasma in the solar atmosphere and convection zone is the key to understanding a large number of solar phenomena and structures. These range from prominences and coronal arches (e.g., Priest 1990) down to photospheric structures (e.g., Lites et al. 1995) or even thick magnetic flux tubes rising in a quasi-stationary regime through the convection zone. Very often, the existing literature has concentrated on cases in which the field has a tubelike structure or in which there was a special symmetry or simplification at hand (force-free field, etc.). A full three-dimensional problem including gravity and nonvanishing magnetic forces and a general energy equation is still too complicated to be tackled analytically, although clear progress has been made in the past decade (see, e.g., references in Low 1993a).

A particular aspect of this problem, the equilibrium and topology of twisted fields (i.e., fields with nonvanishing helicity), is of special relevance at present, since it seems that the magnetic field appearing in the photosphere must have already a nonnegligible net twist in deeper layers in the convection zone (Lites et al. 1995; Tanaka 1991). In the present paper, we study the equilibrium structure of straight horizontal non–force-free magnetic flux tubes with a net amount of twist. We simplify the problem by assuming invariance of all quantities in the direction of the tube axis. The study of the physics of twisted magnetic flux tubes began with Lüst & Schlüter (1954). They, as well as Parker (1974, 1979) and Browning & Priest (1983), considered force-free fields. The effect of gravity has been studied in a general way by Low in his series of papers on magnetostatic atmospheres (1975, 1980, 1981, 1993a, and references therein). For the case of a two-dimensional plasma in a uniform gravity field, Low (1975) showed that the equilibrium can be reduced to a single scalar nonlinear elliptic partial differential equation: the Grad-Shafranov equation. When solving this equation, one should determine the thermodynamics simultaneously with the flux function of the transverse magnetic field (i.e., the components perpendicular to the tube axis). However, this is not easy to achieve because of the nonlinearity of the equation. One possibility is to simplify the thermodynamics by assuming the temperature to be constant. In this case, the Grad-Shafranov equation simplifies into a form (similar to Liouville's equation) which can be integrated (e.g., Dungey 1953; Low 1981; Zweibel & Hundhausen 1982; Webb 1986, 1988). Analytical solutions in which the plasma is assumed to be a polytropic gas have also been found (Lerche & Low 1980; Hundhausen & Low 1994; Low & Hundhausen 1995). Another approach is to choose a priori the magnetic field. The thermodynamic variables are then deduced to match the given magnetic field configuration (e.g., Low 1980; Hu 1988; Lites et al. 1995), but the resulting temperature distribution may be unrealistic. There is also the possibility of choosing the temperature as a nontrivial function of the flux distribution, the latter being obtained afterward by the integration of the Grad-Shafranov equation. In the case of a horizontal magnetic flux tube surrounded by unmagnetized plasma, we are confronted with an additional difficulty: we must solve the equilibrium equations inside a region defined by a *free boundary*, i.e., the shape of the latter must be deduced from the balance of forces. Progress can be achieved by transforming a solution without gravity into a solution with gravity (Dungey 1953; Low 1981; Lites et al. 1995). Other solutions have been found by forcing the boundary of the tube to match an external potential field (Cartledge & Hood 1993; Low 1993b).

In this paper we are interested in obtaining magnetostatic solutions by prescribing as little as possible the transverse magnetic flux function. We want to study how gravity and/or an arbitrary (external) pressure distribution on the boundary determine the shape of the azimuthal field lines. On the other hand, we want to allow for an arbitrary entropy distribution across the tube. Finally, at least for the applications in the convection zone, we are interested in cases in which the cross section of the tube does not span a whole scale height. To find a solution based on those premises, we first write down in § 2 the MHD equations in a generally nonorthogonal coordinate system such that the transverse magnetic field lines are coordinate lines. This transformation to flux coordinates has the advantage of simplifying the boundary condition (e.g., Edenstrasser 1980; Hu 1988; Cally 1991).

To solve the equilibrium equations, we carry out a perturbation analysis which allows solutions with an arbitrary amount of twist in the tube (§ 3). The perturbation scheme used requires only that the ratio of tube radius to pressure scale height is small: it is not a radial expansion of the equations around the tube axis, so that it is much less restrictive than the so-called thin flux-tube approximation (Defouw 1976; Roberts & Webb 1978). The latter requires, in particular, that the azimuthal field component be much smaller than the longitudinal one, whereas our approach does not have this limitation. This is explained in more detail in § 4. In the past, similar linearization techniques have been used successfully to study toroidal coronal loops (Lothian & Hood 1989; Emslie & Wilkinson 1994). They are also related to the expansion techniques used in the fusion literature to study equilibrium configurations for toroidal magnetic fields of small inverse aspect ratio (Greene, Johnson, & Weimer 1971; Miller & Turner 1981).

One of the basic applications of the equations and calculations of the present paper is to twisted flux rings rising slowly enough in the convection zone so that they are permanently in a state of quasi-equilibrium. It can be seen (§ 7) that a sufficient condition for this quasi-static rise is that the radius of the tube be small in comparison to the scale height of the external medium (i.e., the same condition as above). The equilibrium distribution of magnetic and thermodynamic quantities for this free boundary problem including gravity and an arbitrary pressure distribution at the surface is studied in §§ 5 and 6. Finally, § 7 contains a general discussion and conclusions.

2. DESCRIPTION OF THE EQUILIBRIUM

We consider a horizontal magnetic flux tube in static equilibrium with a stratified external medium and concentrate on the case in which there is no variation of the physical quantities along the tube axis. In cylindrical coordinates (r, ϕ, y) with the y-axis along the tube axis and $\phi = 0$ coinciding with the vertical z-axis ($z = 0$ at the center), this means no variation with respect to y (see Fig. 1). We define $\boldsymbol{B}_t$ as the projection of the magnetic field, $\boldsymbol{B}$, on the vertical plane (x, z), and ∇_t as the nabla operator also in the plane (x, z). The equations governing the equilibrium of this system are then

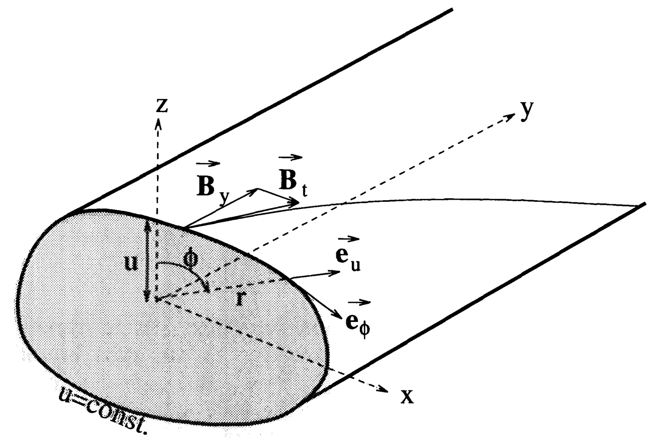

FIG. 1.—Diagram showing the coordinates and some of the symbols used in the paper.

$$0 = \nabla_t \cdot \boldsymbol{B}_t , \quad (1)$$

$$0 = -\nabla_t\left(p + \frac{\boldsymbol{B}^2}{8\pi}\right) + \rho \boldsymbol{G} + \frac{1}{4\pi}(\boldsymbol{B}_t \cdot \nabla_t)\boldsymbol{B}_t , \quad (2)$$

$$0 = \frac{1}{4\pi}[(\boldsymbol{B}_t \cdot \nabla_t)B^y]\boldsymbol{e}_y , \quad (3)$$

$$p_e = p + \frac{\boldsymbol{B}^2}{8\pi} \quad \text{at the boundary} . \quad (4)$$

Here p_e, p, ρ, and $\boldsymbol{G}$ stand for the external and internal gas pressure, density, and gravitational acceleration, respectively; $\boldsymbol{e}_y$ is the unit basis vector in the y-direction. Equations (2) and (3) are the components of the momentum equation in the vertical plane and in the y-direction, respectively. Equation (3) implies that B^y is constant along the lines of force of $\boldsymbol{B}_t$, which, according to equation (1), are closed. The magnetic field thus presents a simple *leeklike structure*: its lines of force are situated on disjoint tubular surfaces characterized by the value of B^y on them, all around a common axis. A further consequence of equation (3) is that the magnetic force vector must lie in the plane (x, z) and, thus, it must be normal to $\boldsymbol{B}_t$.

We will find it more convenient to work with a set of coordinates (u, ϕ, y) instead of (r, ϕ, y), in which u is defined through the requirement that (see Fig. 1) (*a*) it is constant along the lines of force of the azimuthal field $\boldsymbol{B}_t$ and (*b*) it should fulfill $u = r = z$ along the semiaxis $\phi = 0$. We prefer this choice to just choosing the magnetic flux function because of the advantages it entails in dealing with the equations and their solutions, as will become apparent along the text. In this coordinate set, the boundary of the tube coincides with one of the u = const surfaces, $u = U$, say. The set (u, ϕ, y) is generally nonorthogonal and thus more difficult to use than the customary curvilinear coordinates. This disadvantage, though, is offset by the simple form which most of the equations and the boundary conditions adopt in this system. The derivation of the elementary differential operators (divergence, curl, covariant derivative, etc.) in these coordinates is carried out in Appendix A using the techniques of differential geometry (Auslander & Mckenzie 1977; Kobayashi & Nomizu 1963). Here we just write the results of the calculation. The metric is now nondiagonal, with arclength element dl given in the new coordinates by

$$dl^2 = r_{,u}{}^2 du^2 + (r_{,\phi}{}^2 + r^2)d\phi^2 + 2r_{,u}r_{,\phi}\,du\,d\phi + dy^2 . \quad (5)$$

The subscripts $,u$ and $,\phi$ indicate the partial derivatives with respect to u and ϕ, respectively. In the new basis of unit vectors (e_u, e_ϕ, e_y), with e_ϕ pointing in the direction $u =$ const, any arbitrary vector can be written as $v = v^u e_u + v^\phi e_\phi + v^y e_y$. Special care must be taken with the scalar product (we write it with a dot) and with the divergence operator:

$$v \cdot e_u = v^u + v^\phi \frac{r_{,\phi}}{\sqrt{r_{,\phi}{}^2 + r^2}} , \quad (6)$$

$$v \cdot e_\phi = v^u \frac{r_{,\phi}}{\sqrt{r_{,\phi}{}^2 + r^2}} + v^\phi , \quad (7)$$

$$\nabla_t \cdot v = \frac{1}{rr_{,u}}\left[\frac{\partial}{\partial u}(rv^u) + \frac{\partial}{\partial \phi}\left(\frac{rr_{,u}v^\phi}{\sqrt{r_{,\phi}{}^2 + r^2}}\right)\right] . \quad (8)$$

Expressions (6) and (7) correspond to the *orthogonal projection* of any vector v onto the directions of the vectors e_u and e_ϕ, respectively, and, for $r_{,\phi} \neq 0$, they do not coincide with the u- and ϕ-*components* of v. The gradient of any arbitrary scalar f, ∇f, is also more complicated than for cylindrical coordinates, since it contains derivatives with respect to u and ϕ in both the u- and ϕ-components. The exact expression is given in Appendix A (eq. [A4]).

The transverse magnetic field, B_t, has a single component:

$$B_t = B^\phi e_\phi , \quad (9)$$

i.e., $B^u \equiv 0$. Thus, in the coordinates (u, ϕ, y) the magnetic field has just *one* functional degree of freedom; the other degree of freedom allowed by the Maxwell equation (1) has in fact been incorporated into the geometry in the form of the unknown function $r(u, \phi)$.

The simple form of equation (9) permits immediate integration of the solenoidality equation (1) and of the momentum equation in direction e_y, equation (3). Calculating the divergence of the transverse field in equation (9), we find that equation (1) is identical with the requirement that there exists a function $b(u)$ such that

$$B^\phi = \frac{\sqrt{r_{,\phi}{}^2 + r^2}}{rr_{,u}} b(u) , \quad (10)$$

where b has the dimension of a magnetic field. In fact, b is the derivative with respect to u of the transverse magnetic flux $A(u)$ defined by $B_t = -\nabla \times (Ae_y)$. Finally, by calculating its scalar product with e_u and e_ϕ, the momentum equation, equation (2), can be split into the pair of equations

$$0 = -\frac{\partial}{\partial u}\left(p + \frac{B^{y2}}{8\pi} + \frac{b^2}{8\pi}F\right) - \rho G \frac{\partial z}{\partial u} + \frac{b^2}{4\pi} Q , \quad (11)$$

$$0 = -\frac{\partial p}{\partial \phi} - \rho G \frac{\partial z}{\partial \phi} , \quad (12)$$

with

$$F \stackrel{\text{def}}{=} \frac{r_{,\phi}{}^2 + r^2}{r^2 r_{,u}{}^2} , \quad (13)$$

$$Q \stackrel{\text{def}}{=} \frac{\partial F}{\partial \phi} \frac{r_{,u} r_{,\phi}}{2(r_{,\phi}{}^2 + r^2)} - \frac{\sqrt{F}}{R_c} , \quad (14)$$

$$R_c^{-1} \stackrel{\text{def}}{=} \frac{(2r_{,\phi}{}^2 - rr_{,\phi\phi} + r^2)}{(r_{,\phi}{}^2 + r^2)^{3/2}} . \quad (15)$$

G is the modulus of the gravitational acceleration G, R_c is the radius of curvature of the B_t line of force, and the coordinate z is an unknown function of u and ϕ, viz., $z = r(u, \phi) \cos \phi$. From the decomposition of B^ϕ introduced in equation (10), it is apparent that the parameter F contains all the ϕ dependence of the transverse field and thus provides a measure for the degree of convergence or divergence of neighboring azimuthal field lines. The last term in equation (11) is the scalar product of the magnetic tension force $(B_t \cdot \nabla_t)B_t/4\pi$ with the vector $r_{,u}e_u$. Using the transverse magnetic flux $A(u)$ instead of u as independent variable, the "radial" momentum equation (11) transforms into the well-known Grad-Shafranov equation (e.g., Low 1980). Equation (12) tells us that the gas is hydrostatically stratified along the B_t lines of force, which is a direct consequence of the fact that the magnetic force is perpendicular to the surfaces $u =$ const.

Equations (11) and (12), together with the two boundary conditions

$$p_e = p + \frac{B^{y2}}{8\pi} + \frac{b^2}{8\pi} F \quad \text{at } u = U , \quad (16)$$

$$u = r \quad \text{for } \phi = 0 , \quad (17)$$

and the equation of state, constitute a system of equations equivalent to equations (1)–(4). The system of equations (11)–(17) provides two differential equations and two boundary conditions for *five* variables, viz., $p(u, \phi)$, $\rho(u, \phi)$, $B^y(u)$, $b(u)$, and $r(u, \phi)$. Hence, the equilibrium possesses *three* functional degrees of freedom.

For the application of this study to different regions of the solar envelope, we have to keep the possibility open of high and low values of the ratio of gas to magnetic pressure. To analyze the effects of gravity in both regimes of high and low plasma beta, it is then convenient to subtract the external gas stratification from the inside equilibrium. To that end, we define the (gas) pressure and density excess

$$\Delta p(u, \phi) \stackrel{\text{def}}{=} p(u, \phi) - p_e[z(u, \phi)] ,$$
$$\Delta \rho(u, \phi) \stackrel{\text{def}}{=} \rho(u, \phi) - \rho_e[z(u, \phi)] , \quad (18)$$

and we introduce them in the equations using the condition of hydrostatic equilibrium for the external stratification. The resulting equations are identical to equations (11) and (12), with Δp and $\Delta \rho$ substituting for p and ρ, respectively. The importance of the gravity force in modifying the equilibrium can be quantified with the dimensionless parameter

$$\epsilon \stackrel{\text{def}}{=} \frac{U}{H_e} , \quad (19)$$

where H_e is the external pressure scale height. If equation (19) is vanishingly small, there will be no variation of the external pressure along the tube boundary, and the equilibrium can be axisymmetric. On the other hand, if it is nonnegligible, some of the forces in the tube must be ϕ dependent so that the external pressure variation can be compensated along the tube boundary. This can be best seen by integrating the momentum equation (2) over any arbitrary region $\mathscr{D}$ delimited by a line $u = \text{const}$,

$$\int_{\partial\mathscr{D}} \left(\Delta p + \frac{b^2}{8\pi} F\right) \boldsymbol{n}\, dl_\phi - \boldsymbol{G} \iint_{\mathscr{D}} \Delta\rho\, d\Sigma = 0 , \qquad (20)$$

where dl_ϕ is the elementary arc length along a line $u = \text{const}$, $\boldsymbol{n}$ is the outward unit normal vector at the point (u, ϕ), $d\Sigma$ is the infinitesimal surface element, and the fact that $B^y = \text{const}$ on $\partial\mathscr{D}$ has been used. We see that the integrand on the left-hand side has to reorganize itself along $\partial\mathscr{D}$ to compensate for the buoyancy force acting on the region $\mathscr{D}$. To be more precise, it is the transverse magnetic field which must adapt its ϕ dependence to counteract the density difference $\Delta\rho$ and achieve the equilibrium: Δp is fixed by the hydrostatic condition in equation (12). Vice versa, the intensity of $\boldsymbol{B}_t$ determines the spacing of the lines $u = \text{const}$ and, hence, it has a direct influence on the "radial" profile of $\Delta\rho$. If we expand the region $\mathscr{D}$ to the whole tube section, the line integral in equation (20) disappears because the total pressure excess is zero at the boundary, and there remains the integral condition

$$0 = \boldsymbol{G} \int_0^{2\pi} \int_0^U \Delta\rho\, r r_{,u}\, du\, d\phi , \qquad (21)$$

which ensures that the total buoyancy force is zero.

Thus, the azimuthal dependence of the variables in the tube plays a central role in the problem. In fact, later on in the paper it will prove useful to use, instead of the "radial" momentum equation (11) and the boundary condition (16), their ϕ derivatives. We obtain therefore the equivalent system to equations (11)–(17), namely,

$$0 = G\left(\frac{\partial \Delta\rho}{\partial u}\frac{\partial z}{\partial \phi} - \frac{\partial \Delta\rho}{\partial \phi}\frac{\partial z}{\partial u}\right) - \frac{\partial}{\partial u}\left(\frac{b^2}{8\pi}\frac{\partial F}{\partial \phi}\right) + \frac{b^2}{4\pi}\frac{\partial Q}{\partial \phi} , \qquad (22)$$

$$0 = -\frac{\partial \Delta p}{\partial \phi} - \Delta\rho G \frac{\partial z}{\partial \phi} , \qquad (23)$$

$$0 = \frac{\partial \Delta p}{\partial \phi} + \frac{b^2}{8\pi}\frac{\partial F}{\partial \phi} \quad \text{at } u = U , \qquad (24)$$

$$u = r \quad \text{for } \phi = 0 , \qquad (25)$$

Equation (22) follows from the introduction of equation (12) into the ϕ-derivative of equation (11). To avoid the loss of information associated with the partial derivatives, we must consider, in addition, the restriction of equations (11) and (16) to the vertical semiaxis $\phi = 0$. With these two additional conditions, the system of equations (22)–(25) is equivalent to the system of equations (11)–(17).

Using equations (23) and (24), it is possible to express in an exact way the deformation of the tube surface in terms of the density excess, the plasma beta of the transverse field, and the transverse size of the tube (as measured by the local scale height):

$$\frac{\partial F}{\partial \phi} = \frac{8\pi p_e}{b^2}\frac{\Delta\rho}{\rho_e}\frac{U}{H_e}\frac{\partial}{\partial \phi}\left(\frac{z}{U}\right) \quad \text{at } u = U . \qquad (26)$$

If, for example, $\Delta\rho$ were positive in the outer tube layers (this must happen, e.g., if the core of the tube is buoyant), then equation (26) indicates that the $\boldsymbol{B}_t$ field lines must be closer together in the upper part of the cross section of the tube than at the bottom (see also Fig. 2 in § 3). The opposite applies if $\Delta\rho < 0$ along the boundary (see Fig. 3 in § 3). More generally, equation (26) shows that a stronger deformation of the tube will be obtained for a smaller value of the transverse magnetic field, a greater value of $|\Delta\rho/\rho_e|$, and/or a larger radius of the tube as compared to the external pressure scale height. From equation (26), we can also immediately obtain an expression for the variation of the pitch angle of the field lines:

$$\frac{\partial \Psi}{\partial \phi} = \frac{8\pi p_e}{(B^y)^2}\frac{\Delta\rho}{\rho_e}\frac{U}{H_e}\frac{\partial}{\partial \phi}\left(\frac{z}{U}\right) \quad \text{at } u = U , \qquad (27)$$

where Ψ is the square of the trigonometric tangent of the pitch angle.

Until now, the only restriction imposed on our model has been the horizontal (or y-) invariance. This has allowed us to eliminate two of the initial four differential equations. However, as explained in the introduction, the resulting system of equations (22)–(24) is still too complicated to solve analytically without some additional simplification. In the following, we consider the gravitation as a perturbation in the equilibrium equations. Using this approximation, we will be able to linearize and integrate the equations.

3. PERTURBATION SOLUTION FOR A HORIZONTAL TUBE IN STATIC EQUILIBRIUM WITH A STRATIFIED EXTERNAL MEDIUM

In this section, we look for solutions to the system of equations (22)–(25) in the limit in which the gravitational terms in the equations can be considered a perturbation to the equilibrium. Considering the parameter ϵ defined in equation (19), this approximation is equivalent to the condition

$$\epsilon \ll 1 , \qquad (28)$$

i.e., small tube radius as measured by the local scale height. This limit is easily applicable to magnetic flux tubes located in the convection zone (except in its uppermost layers, e.g., Moreno-Insertis 1992) and for the magnetic flux arcades in the corona (e.g., Priest 1982). We carry out a perturbation analysis by considering, first, an *axisymmetric* magnetic flux tube in equilibrium with a background plasma of constant pressure ($\boldsymbol{G} = 0$), and second, the same equilibrium plus a linear correction attributable to gravity. For the sake of simplicity, we do not consider nonaxisymmetric equilibria in the absence of gravity.

3.1. *Linearization of the Equations*

We define for each physical quantity a zero- and a first-order symbol:

$$\begin{aligned} p &= p_0(u)[1 + p_1(u, \phi)] & p_e &= p_{e0}[1 + p_{e1}(u, \phi)] , \\ \rho &= \rho_0(u)[1 + \rho_1(u, \phi)] & \rho_e &= \rho_{e0}[1 + \rho_{e1}(u, \phi)] , \\ \Delta p &= \Delta p_0(u)[1 + \Delta p_1(u, \phi)] & \Delta\rho &= \Delta\rho_0(u)[1 + \Delta\rho_1(u, \phi)] , \\ B^y &= B^y_0(u)[1 + B^y_1(u)] & B^\phi &= B^\phi_0(u)[1 + B^\phi_1(u, \phi)] , \\ b &= b_0(u)[1 + b_1(u)] & r &= u[1 + r_1(u, \phi)] . \end{aligned} \qquad (29)$$

The zero-order variables correspond to the unperturbed axisymmetric tube, whereas the first-order terms are linear in the gravitational acceleration G (equivalently, in $\epsilon \overset{\text{def}}{=} U/H_{e0}$). Introducing equation (29) into equation (10) and neglecting the second-order terms, we obtain

zero order:

$$B_0^\phi = b_0 ; \tag{30}$$

first order:

$$B_1^\phi = b_1 - \frac{\partial}{\partial u}(ur_1) . \tag{31}$$

To *zero order* ($G = 0$), the external pressure is constant, and we have assumed that the tube is axisymmetric. Thus, from the equilibrium equations (22)–(25) there follows the customary radial momentum equation and boundary condition describing the equilibrium of an axisymmetric flux tube

$$0 = -\frac{\partial}{\partial u}\left(\Delta p_0 + \frac{B_0^{y2}}{8\pi} + \frac{b_0^2}{8\pi}\right) - \frac{b_0^2}{4\pi}\frac{1}{u} , \tag{32}$$

$$0 = \Delta p_0 + \frac{B_0^{y2}}{8\pi} + \frac{b_0^2}{8\pi} \quad \text{at } u = U , \tag{33}$$

whereas to *first order* equations (22)–(25) transform into

$$0 = -\frac{\partial \Delta\rho_0}{\partial u} Gu \sin\phi + \frac{\partial}{\partial u}\left[\frac{b_0^2}{4\pi}\frac{\partial}{\partial u}(ur_{1,\phi})\right] + \frac{b_0^2}{4\pi}\frac{1}{u}\left[\frac{\partial}{\partial u}(ur_{1,\phi}) + r_{1,\phi} + r_{1,\phi\phi\phi}\right] , \tag{34}$$

$$0 = -\Delta p_0 \frac{\partial \Delta p_1}{\partial \phi} + \Delta\rho_0 Gu \sin\phi , \tag{35}$$

$$0 = \Delta p_0 \frac{\partial \Delta p_1}{\partial \phi} - \frac{b_0^2}{4\pi}\frac{\partial}{\partial u}(ur_{1,\phi}) \quad \text{at } u = U , \tag{36}$$

$$0 = r_1 \quad \text{for } \phi = 0 . \tag{37}$$

Finally, the integral condition (21) yields, up to first order,

$$0 = G \int_0^U \Delta\rho_0 \, u \, du . \tag{38}$$

Two useful quantities which will be used later on in the paper are the total pressure and the total pressure excess at an arbitrary point in the tube, viz.,

$$\mathscr{P} \overset{\text{def}}{=} p + \frac{B^{y2}}{8\pi} + \frac{B^{\phi 2}}{8\pi} , \quad \Delta\mathscr{P} \overset{\text{def}}{=} \mathscr{P}(u, \phi) - p_e[z(u, \phi)] , \tag{39}$$

and their zero- and first-order values $\mathscr{P}_0$, $\mathscr{P}_1$, $\Delta\mathscr{P}_0$, and $\Delta\mathscr{P}_1$.

3.2. *Solutions to the Zero- and First-Order Equations*

To zero order we have only one differential equation and one boundary condition for the four unknowns $\Delta p_0(u)$, $\Delta\rho_0(u)$, $b_0(u) = B_0^\phi(u)$, and $B_0^y(u)$. Thus, the nonperturbed tube has 3 functional degrees of freedom. A general solution to the momentum equation (32) and its boundary condition (33) can be obtained by choosing two among the following four functions, $b_0(u)$, $\Delta\mathscr{P}_0(u)$, $B_0^y(u)$, and $\Delta p_0(u)$, and obtaining the remaining two from equation (32) in the form (see Lüst & Schlüter 1954)

$$\Delta p_0 + \frac{B_0^{y2}}{8\pi} = \frac{1}{2u}\frac{\partial}{\partial u}(u^2 \Delta\mathscr{P}_0) , \tag{40}$$

and the definition in equation (39). The choice of the two functions is arbitrary except for the minimal requirement that all the pressure values (gas and magnetic) derived from equation (40) be everywhere positive, and that the boundary condition in equation (33) be fulfilled. As seen below (calculation of r_1), it may be necessary to impose further restrictions on the zero-order variables to avoid singularities in the first-order equilibrium.

The last function to be determined, the density excess $\Delta\rho_0(u)$ [alternatively, the temperature excess $\Delta T_0(u)$], can be chosen freely subject to the integral condition in equation (38). As explained below for the first-order variables, the thermodynamics of rising tubes in the convection zone helps in choosing a reasonable value for $\Delta\rho_0(u)$ (see § 3.2.3 later on in this subsection).

Once the zero-order equations are solved, the first-order system of equations (34)–(37) provides two differential equations and two boundary conditions for the five unknown functions $\Delta p_1(u, \phi)$, $\Delta\rho_1(u, \phi)$, $b_1(u)$, $B_1^y(u)$, and $r_1(u, \phi)$. Therefore, the first-order system also has 3 functional degrees of freedom. This threefold freedom can be considerably restricted, however. We will discuss the solution for the five variables separated in three groups.

3.2.1. $r_1(u, \phi)$ *and* $\Delta p_1(u, \phi)$: *The* ϕ *Dependence of the Problem*

Solving for these variables will allow us to eliminate most of the ϕ dependence from the problem. First of all, the momentum equation in direction e_ϕ, equation (35), can be integrated immediately to give

$$\Delta p_0 \Delta p_1 = \Delta p_0 \Delta p_1^* + \Delta\rho_0 Gu(1 - \cos\phi) , \tag{41}$$

where the superscript "*" means restriction of a function to the semiaxis $\phi = 0$, i.e.,

$$\Delta p_1^* \overset{\text{def}}{=} \Delta p_1(u, \phi = 0) . \tag{42}$$

Concerning $r_1(u, \phi)$, the requirement of 2π periodicity in ϕ together with the form of equation (34) and the boundary conditions (36) and (37) strongly suggest a solution with the same ϕ dependence as equation (41), i.e.,

$$r_1(u, \phi) \overset{\text{def}}{=} a(u)(\cos\phi - 1) . \tag{43}$$

Introducing equation (43) into the momentum equation (34) completely eliminates the ϕ dependence. Two subsequent integrations of that equation with respect to u yield the solution for $a(u)$:

$$a(u) = -\frac{1}{u}\int_0^u \frac{4\pi}{b_0^2}\frac{G}{u'}\left(\int_0^{u'} \frac{\partial \Delta\rho_0}{\partial u''} u''^2 \, du''\right) du' . \tag{44}$$

Looking in retrospect, the zero order variables $\Delta\rho_0$ and b_0 must be such that a nonsingular $a(u)$ results: as discussed in Appendix B, this implies a condition on the approach to zero for the function $b_0(u)$ in the neighborhood of the tube axis.

Given the linearity of the equilibrium equations (34)–(37), the difference between two sets (Δp_1, $\Delta\rho_1$, B_1^y, b_1, r_1) of solutions to those equations is a solution for the case $G = 0$. Consequently, equations (43) and (44) provide a *unique* solution for r_1 in the sense that any other solution must be equal to it modulo a nonaxisymmetric solution to the problem without gravity.

Equation (41) solves for the ϕ dependence of $\Delta p_1(u, \phi)$ but leaves its u dependence [i.e., essentially $\Delta p_1^*(u)$] undetermined. The latter will be considered together with the remaining variables in the following section.

3.2.2. $\Delta p_1^*(u)$, $b_1(u)$, $B_1^y(u)$

Two of these three functions can be chosen freely. The third is then determined by the restriction of the radial momentum equation, equation (11), and of the boundary condition in equation (16), to the semiaxis $\phi = 0$; similarly to equation (40),

$$\Delta p_0 \Delta p_1^* + \frac{2B_0^{y2} B_1^y}{8\pi} = \frac{1}{2u}\frac{\partial}{\partial u}(u^2 \Delta\mathscr{P}_0 \Delta\mathscr{P}_1^*) + \frac{u}{2}\left[\Delta\rho_0 G + \frac{b_0^2}{4\pi}\frac{a(u)}{u}\right] . \quad (45)$$

3.2.3. $\Delta\rho_1(u, \phi)$: *Thermodynamical Considerations*

The choice of $\Delta\rho_1$ (or, equivalently, ΔT_1) completes the solution to the first-order system of equations (34)–(37). The freedom of choice of this variable can be restricted by imposing a condition on the temperature or entropy distribution in the tube. For tubes that rise through the convection zone, for instance, it is reasonable to assume that the entropy stays constant, since the timescale of rise is much shorter than the time for thermal exchange with the surroundings (Moreno-Insertis 1983). Thus, for a tube which comes from the deep convection zone where ϵ is very small, we can assume the entropy s to be *constant* in the interior. Thus, $s_0 = \text{const}$ and $s_1 = 0$. Consequently, $\Delta\rho/\rho_e$ can be given as a function of $\Delta p/p_e$ (this also applies to the zero-order variables).

Summarizing the results for the zeroth and first-order solution, once a choice has been made for the distribution of the thermodynamical variable s (or T) in the tube and using the ϕ dependence given in equations (41) and (43), the equilibrium can be determined by choosing two functions of the triplet $[\Delta p_0, b_0(u), B_0^y(u)]$, for the zero order, and another two from the set $[\Delta p_1^*(u), b_1(u), B_1^y(u)]$ for the first order. Further restrictions on this freedom can be imposed by requiring that the magnetic flux of the perturbed and unperturbed equilibria be the same.

To illustrate the general solution given in this subsection and our considerations in the next subsection, we have calculated in Appendix B the equilibrium resulting when $B_0^y(u)$, $B_1^y(u)$, $b_0(u)$, and $b_1(u)$ have power-law profiles. In Figures 2 and 3, for the sake of simplicity, we just plot the particular equilibria obtained when

$$\frac{b_0^2}{8\pi p_{e0}} \overset{\text{def}}{=} \alpha_\phi \left(\frac{u}{U}\right)^{2m_\phi} , \quad b_1 \overset{\text{def}}{=} 0 , \quad (46)$$

$$\frac{B_0^{y2}}{8\pi p_{e0}} \overset{\text{def}}{=} \alpha_y + \xi\left[\left(\frac{u}{U}\right)^{2m_y} - \frac{1}{m_y + 1}\right] , \quad B_1^y \overset{\text{def}}{=} 0 , \quad (47)$$

where α_ϕ, α_y, ξ, m_ϕ, and m_y are constants ($m_\phi \geq 1$ to avoid an infinite current density along the axis of the tube). Since $r_1 \neq 0$, the resulting tube is nonaxisymmetric.

3.3. *Amplitude of the Deformation and the Intensity of* B_t

Using the results of the previous sections, we can study in detail the equilibrium of the plasma *inside* the tube. In what follows, we shall put particular emphasis on the momentum equation (34), which indicates how the shape of the tube depends on the profiles of $b_0(u)$ and $\Delta\rho_0(u)$.

Up to first order, the radius of curvature R_c and the geometric factor F, defined in equation (13), are

$$F \cong 1 + F_1 = 1 - 2\frac{\partial}{\partial u}(ur_1) ,$$

$$\frac{1}{R_c} \cong \frac{1}{R_{c0}}(1 - R_{c1}) = \frac{1}{u}(1 - r_1 - r_{1,\phi\phi}) . \quad (48)$$

Inserting $r_1 = a(u)(\cos\phi - 1)$ into equation (48), one obtains that the radius of curvature R_c is ϕ independent. Further, the latter is an intrinsic quantity, i.e., it is independent of the particular coordinate set being used. Thus, *the azimuthal lines of force are circles* in this first-order approximation (Fig. 2). The ϕ dependence of the magnetic tension, necessary to compensate for the effect of the gravitation, is therefore achieved by simply shifting vertically the geometrical centers of the azimuthal field lines with respect to their zero-order location (see Figs. 2*a* and 3*a*).

In § 2 we have seen that the equilibrium along a $\boldsymbol{B}_t$ field line depends on both $\Delta\rho$ and the convergence or divergence of the azimuthal field lines as measured by the parameter F. With the linearized equations in hand, it is now possible to write the following simple relation for F *inside* the tube:

$$F_1 = \frac{8\pi p_{e0}}{b_0^2}\frac{\Delta\rho_0 - \overline{\Delta\rho_0}}{\rho_{e0}}\epsilon\left[\frac{u}{U}(\cos\phi - 1)\right] . \quad (49)$$

The horizontal bar indicates an area average of the given function, viz.,

$$\bar{f}(u) \overset{\text{def}}{=} \frac{1}{u^2}\int_0^u f(u')2u'\,du' . \quad (50)$$

In equation (49) $\Delta\rho_0$ is related to the variation of the gas pressure excess along the line $u = \text{const}$, and $\overline{\Delta\rho_0}$ corresponds to the average buoyancy force. Therefore, the structure of the azimuthal field lines is directly related to $\Delta\rho_0(u) - \overline{\Delta\rho_0}(u)$, i.e., to the *differential buoyancy* in the tube interior.

As an illustration to the interdependence between F_1, $\Delta\rho_0$, and $\overline{\Delta\rho_0}$, consider the equilibrium along the line $u = 0.6$ in Figure 2 (marked by little plus signs). The plasma enclosed by this particular azimuthal field line is buoyant relative to the matter located on the field line itself: $\Delta\rho_0 > \overline{\Delta\rho_0}$. To counteract this buoyancy, the total pressure excess must increase with height along $u = 0.6$ (Fig. 2*f*). This can only be achieved through the increase of the magnetic pressure of the azimuthal field (Fig. 2*c*), and that is why the field lines have to converge toward higher levels, i.e., F increases with height, as required by equation (49). In Figure 3 we show an equilibrium similar to that of Figure 2 but with the sign of the radial gradients of $B^{y2}/8\pi p_{e0}$ and $\Delta\rho/\rho_{e0}$ reversed. In this case, the plasma enclosed by the field line $u = 0.6$ feels a net negative buoyancy relative to the magnetic field line considered: $\Delta\rho_0 - \overline{\Delta\rho_0} < 0$. Therefore, the azimuthal field lines must separate from each other with height, so that the original circles shift toward the bottom of the tube.

The deviation from axisymmetry will be larger the greater the first three terms on the right-hand side of equation (49). A sufficient condition for this deformation not to be too strong (so that F_1 remains of first order) is, then, that

$$\left|\frac{p_{e0}}{b_0^2}\frac{8\pi}{}\frac{\Delta\rho_0 - \overline{\Delta\rho_0}}{\rho_{e0}}\right| \lesssim 1 \quad (51)$$

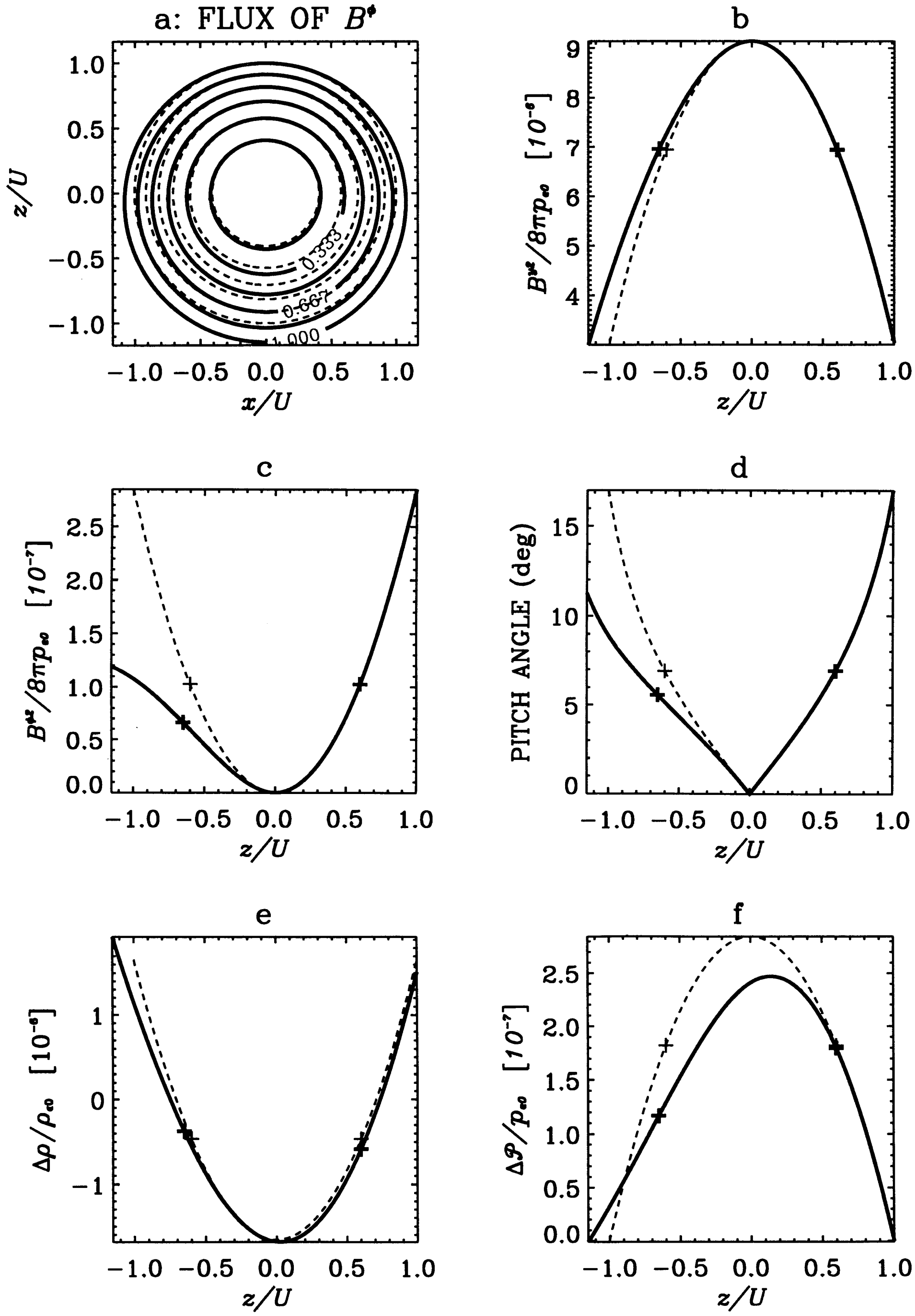

FIG. 2.—Horizontal magnetic flux tube in static equilibrium with a stratified polytropic atmosphere which simulates the convection zone at a depth of 200,000 km. The first plot represents the isolines of the transverse magnetic flux normalized to its boundary value. The other plots are the profiles of (*b*) $B^{y2}/8\pi p_{e0}$, (*c*) $B^{\phi 2}/8\pi p_{e0}$, (*d*) arctan (B^ϕ/B^y), (*e*) $\Delta\rho/\rho_{e0}$, and (*f*) $\Delta\mathscr{P}/p_{e0}$ along the vertical *z*-axis (*solid lines*). The zero-order profiles are overplotted (*dashed lines*). The plus signs are located at the intersections between the line $u = 0.6$ and the *z*-axis. The algebraic form of the zero- and first-order variables plotted here are laid out in Appendix B (with zero substituted for b_1 and B_1^y, and $\alpha_\phi = 3 \times 10^{-7}$, $\alpha_y = 6 \times 10^{-6}$, $\xi = -6 \times 10^{-6}$, $m_\phi = 1$, and $m_y = 1$).

in the whole tube. In Figures 2 and 3, for illustration purposes, we have chosen an equilibrium for which F_1 is only marginally of first order. Equation (51) sets a lower bound for the transverse magnetic field intensity: b must be strong enough for the change resulting from gravity to be no more than a *perturbation* of the zero-order axisymmetric equilibrium. An upper bound to b is given by the condition that the transverse magnetic field intensity is not above the threshold for the onset of the kink instability (see the discussion in § 7).

4. COMPARISON WITH THE SLENDER FLUX-TUBE APPROXIMATION

In § 3 we have used a perturbation analysis to solve the system of equations (22)–(25) obtained in § 2. Another pertur-

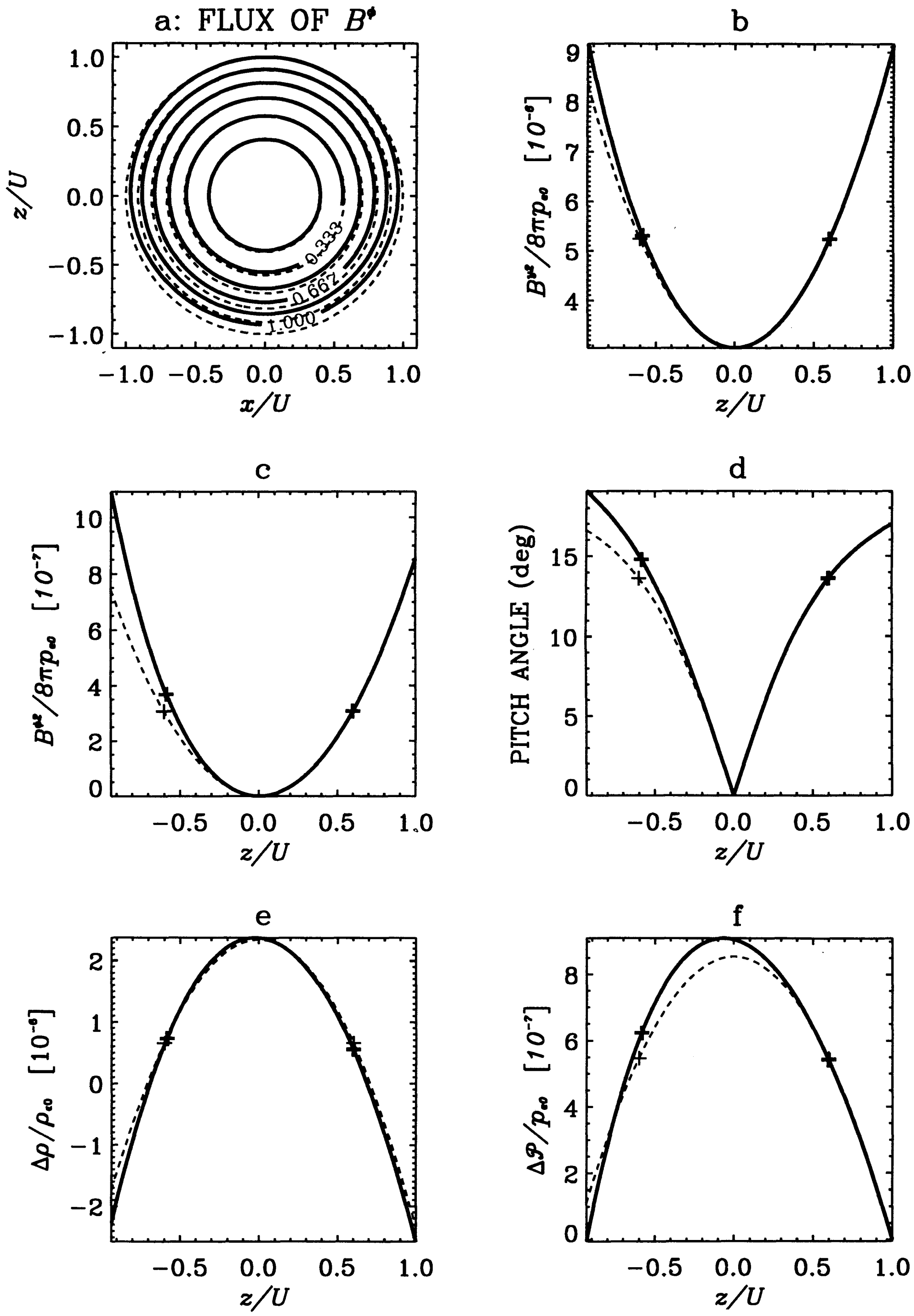

FIG. 3.—Same as Fig. 2, but now with radial gradients of $B^{y2}/8\pi p_{e0}$ and $\Delta\rho/\rho_{e0}$ with reversed sign (but the same absolute value). The core of the tube is now heavier than the outer layers. The parameters of the equilibrium are the same as in Fig. 2, except for the sign of ξ and for the value of α_ϕ, which is now 9×10^{-7}.

bation scheme used in the literature and applied widely to calculations of magnetic flux-tube evolution is the so-called *slender flux-tube approximation*, which consists of using truncated Taylor expansions of the radial variation of all quantities in the tube (e.g., Roberts & Webb 1979; Browning & Priest 1983; Feriz-Mas & Schüssler 1989). This approximation is valid as long as the radius of the tube is much smaller than the length scale of variation of all quantities inside and outside the tube. In the case of a horizontal tube embedded in a stratified atmosphere considered here, the slender flux-tube approximation would rest on the following power expansions (we use different symbols for the indices to avoid confusion with previous sections):

$$
\begin{aligned}
r &= r_{(1)}(u/H_e) + r_{(2)}(u/H_e)^2 + \cdots ,\\
b &= b_{(1)}(u/H_e) + b_{(2)}(u/H_e)^2 + \cdots ,\\
F &= F_{(0)} + F_{(1)}(u/H_e) + F_{(2)}(u/H_e)^2 + \cdots ,
\end{aligned}
\tag{52}
$$

and similarly for the other variables, with the coefficients $f_{(k)}$ for any function f given by the derivatives of f on the axis in an elementary way. Although apparently similar, there are important differences between the perturbation approach of the previous sections and the slender flux-tube approximation. As pointed out by Browning & Priest (1983), the use of the Taylor expansions implicitly requires that the transverse field B^ϕ be small compared to the longitudinal field B^y: B^ϕ must be at least of first order in the expansion parameter U/H_e, whereas B^y is of zero order. This is not the case in our perturbation scheme: we are not constraining the range of values of B^ϕ/B^y. The only limit is given by instability considerations, as explained elsewhere in this paper.

The use of the Taylor expansions also has restrictive consequences on the structure of the density excess in the tube. This can be easily seen by expanding the equations of § 2 with the help of equation (52). We show the difference in the case of the boundary condition equation (24) obtained in § 2. The expansion in u/H_e of this equation leads to

order ϵ^1:

$$0 = \Delta\rho_{(0)} ; \tag{53}$$

order ϵ^2:

$$\frac{dF_{(0)}}{d\phi} = \frac{8\pi p_{e(0)}}{b_{(1)}^2} \frac{\Delta\rho_{(1)}}{\rho_{e(0)}} \frac{U}{H_e} \frac{d}{d\phi}\left(\frac{r_{(1)}}{U} \cos\phi\right) , \tag{54}$$

at the boundary. Thus, in addition to the inequality

$$\left| \frac{8\pi p_{e(0)}}{b_{(1)}^2} \frac{\Delta\rho_{(1)}}{\rho_{e(0)}} \right| \lesssim \mathcal{O}(\epsilon^0) , \tag{55}$$

which is similar in form to equation (51), the use of the Taylor expansion implies that

$$|\Delta\rho| \lesssim \mathcal{O}(\epsilon^1) \tag{56}$$

in the *whole* tube, whereas our perturbation study requires only that

$$|\overline{\Delta\rho}(U)| \lesssim \mathcal{O}(\epsilon^1) . \tag{57}$$

Summarizing, the slender flux-tube approximation is a more restrictive expansion procedure than the perturbation scheme used in this paper. The former can, in general, be applied only to regions close enough to the axis of a tube. This limitation has several consequences as to (at least) the degree of twisting and the density distribution in the interior of the tube.

5. PERTURBATION CAUSED BY AN ARBITRARY EXTERNAL PRESSURE DISTRIBUTION

In §§ 2 and 3 we have studied the modifications caused by gravity in the equilibrium structure of a twisted horizontal flux tube. With a view to studying flux tubes in relative motion to their surroundings, we consider in this section the deformation caused by an arbitrary pressure fluctuation along the boundary of the tube, $p_{ef}(\phi)$ (as caused by, e.g., a flow pattern around the tube), in the absence of gravity. The total external pressure in this case is of the form $p_e + p_{ef}(\phi)$, with $p_e = \text{const}$. The deformation of the tube will be carried about via the boundary condition instead of through a body force, as before.

For simplicity of notation, we define here the pressure and density excess in the tube using again equation (18), although now p_e is no longer a function of $z(u, \phi)$, and it is only part of the total external pressure. The equilibrium in the present problem is governed by the following system of equations:

$$0 = -\frac{\partial}{\partial u}\left(\frac{b^2}{8\pi}\frac{\partial F}{\partial\phi}\right) + \frac{b^2}{4\pi}\frac{\partial Q}{\partial\phi} , \tag{58}$$

$$0 = -\frac{\partial\Delta p}{\partial\phi} , \tag{59}$$

$$\frac{\partial p_{ef}}{\partial\phi} = \frac{b^2}{8\pi}\frac{\partial F}{\partial\phi} \quad \text{at } u = U , \tag{60}$$

$$u = r \quad \text{for } \phi = 0 . \tag{61}$$

Like equations (22) and (24) of § 2, equations (58) and (60) result from the ϕ derivatives of the "radial" momentum equation and of the boundary condition, respectively. Thus, they must be completed with the restriction of the "radial" momentum equation and of the boundary condition $\Delta\mathscr{P}(U, \phi) = p_{ef}(\phi)$ to the vertical semiaxis $\phi = 0$. As a consequence of the absence of gravity, $\Delta\rho$ drops out of the equilibrium equations: we now have only *four* unknown functions (Δp, B^y, b, and r) for the same number of equations.

The momentum equation in direction $\boldsymbol{e}_\phi$, equation (59), indicates that Δp, like B^y, is ϕ invariant: apart from the gas pressure gradient, there is no force along the $\boldsymbol{B}_t$ field lines. Therefore, the only variable which can vary along ϕ to achieve the equilibrium is r, and consequently, B^ϕ (see eq. [10]). The transverse magnetic field must then organize itself to counteract the pressure fluctuations along the boundary (eq. [60]) and to sustain its own ϕ dependence inside the tube (eq. [58]). As in § 2, the boundary condition can be transformed into an exact expression for the variation of the mutual proximity of the transverse magnetic field lines:

$$\frac{\partial F}{\partial\phi} = \frac{\partial}{\partial\phi}\left(\frac{8\pi p_{ef}}{b^2}\right) \quad \text{at } u = U . \tag{62}$$

We see that the deformation of the magnetic pattern depends on the intensity of $\boldsymbol{B}_t$: only if $b^2/8\pi \gg |p_{ef}|$ will the tube be able to withstand the external perturbation with only minor deformation. We will come back on this point in the discussion (§ 7).

In order to have a static equilibrium, the net force exerted by the total pressure along any $\boldsymbol{B}_t$ line of force must be zero. Applied to the line $u = U$, this condition implies that

$$0 = -\int_{\partial\mathscr{D}} p_{ef}\, \boldsymbol{n}\, dl_\phi , \tag{63}$$

i.e., the total force exerted by p_{ef} must cancel. One can only relax this condition in the presence of gravity: see § 6.

5.1. *Perturbation Analysis*

If $|p_{ef}|/p_e \ll 1$, we can perform a perturbation analysis following the scheme of § 3: we consider, to zero order, an axisymmetric tube in equilibrium with an external medium of constant pressure $p_{e0} = p_e$. The perturbation caused by p_{ef} is then taken into account with the first-order variables; $|p_{ef}|/p_e \ll 1$ is a good approximation whenever p_{ef} is caused by external subsonic flows.

We have given p_{ef} the following general form:

$$p_{ef}(\phi) \stackrel{\text{def}}{=} \left[\frac{b_0^2(U)}{8\pi}\right] P(\phi) \quad \text{with}$$

$$P(\phi) = P_0 + \sum_{k=2}^{\infty} [P_{ck}\cos(k\phi) + P_{sk}\sin(k\phi)] , \tag{64}$$

and we have excluded the $k = 1$ terms so that no net pressure force results (as required by eq. [63]). The deformation of the tube shape caused by the external pressure fluctuation p_{ef} can then be quantified with the dimensionless parameter

$$\epsilon_f = \max_\phi [|P(\phi)|] . \tag{65}$$

With these definitions in hand, we can linearize our system of equations, (58)–(61). The zero-order equilibrium is the same as in § 3. The only difference is that, now, the mean density excess in the tube need not necessarily be zero. To first order, we obtain

$$0 = \frac{\partial}{\partial u}\left[\frac{b_0^2}{4\pi}\frac{\partial}{\partial u}(ur_{1,\phi})\right] + \frac{b_0^2}{4\pi}\frac{1}{u}\left[\frac{\partial}{\partial u}(ur_{1,\phi}) + r_{1,\phi} + r_{1,\phi\phi\phi}\right], \tag{66}$$

$$0 = \frac{\partial \Delta p_1}{\partial \phi}, \tag{67}$$

$$\frac{\partial P}{\partial \phi} = -2\frac{\partial}{\partial u}(ur_{1,\phi}) \quad \text{at } u = U, \tag{68}$$

$$0 = r_1 \quad \text{for } \phi = 0. \tag{69}$$

For each set $(\Delta p_0, B_0^y, b_0)$ of solutions to the zero-order system, the most general function $r_1(u, \phi)$ which can be a solution of equation (66) and satisfy the boundary conditions (68) and (69) has the form

$$r_1 = a_0(u) + \sum_{k=2}^{\infty}[a_{ck}(u)\cos(k\phi) + a_{sk}(u)\sin(k\phi)]. \tag{70}$$

The introduction of equation (70) in equation (66) permits elimination of the ϕ dependence and leads to the following linear homogeneous differential equation:

$$0 = \frac{\partial}{\partial u}\left[\frac{b_0^2}{4\pi}\frac{\partial}{\partial u}(ua_k)\right] + \frac{b_0^2}{4\pi}\frac{1}{u}\left[\frac{\partial}{\partial u}(ua_k) + (1-k^2)a_k\right], \tag{71}$$

for each $a_k = a_{ck}, a_{sk}$, and $k = 2, 3, 4, \ldots$, independently of the others. Similarly to what happens in § 3, the nonsingularity condition for the a_k values in the center of the tube may restrict somewhat the functional freedom of b_0. Once equation (71) is solved, equation (70) yields r_1. The other unknowns $B_1^y(u)$, $b_1(u)$, and $\Delta p_1(u)$, are determined in a similar way as in § 3.2.2: two of these three functional degrees of freedom can be freely chosen; the third follows from the restriction of the "radial" momentum equation and of the boundary condition to the vertical semiaxis $\phi = 0$.

To illustrate the general process described above, we have calculated in Appendix C a class of solutions to the present equilibrium problem with power-law distributions of the zero- and first-order variables similar to those of Appendix B. For comparison with Figures 2 and 3, we plot in Figure 4 the equilibrium configuration that is obtained for the particular case $B_1^y = b_1 = 0$ (see the definitions in eqs. [46] and [47]) and $P = P_{c3}\cos(3\phi)$.

5.2. *The Shape of the Tube*

Integration of the boundary condition in equation (68) gives

$$F_1(\phi) = P(\phi) - P(\phi = 0), \tag{72}$$

so that, at the boundary, p_{ef} completely determines the transverse magnetic field pattern. Inside the tube, the state of affairs is more complicated because $\boldsymbol{B}_t$ must sustain its own variation with respect to ϕ: the magnetic *tension* must vary in order to compensate for the magnetic *pressure gradient* associated with $\boldsymbol{B}_t$ (see eq. [58]). Thus, unlike what happens to a tube perturbed by gravity, the azimuthal magnetic field lines will have to deviate from a circular shape in order to maintain the equilibrium.

An exact expression for the relation between the variation of the mutual proximity of the field lines of $\boldsymbol{B}_t$ and their radius of curvature can be obtained by integrating equation (66) over the surface enclosed by a line u = const:

$$\frac{b_0^2}{u}\frac{\partial F_1}{\partial \phi} = \overline{\frac{b_0^2}{u}\frac{\partial R_{c1}}{\partial \phi}}(u) \tag{73}$$

($R_{c1} = r_1 + r_{1,\phi\phi}$ is the correction to the zero-order radius of curvature u). The variation of $B^{\phi 2}/8\pi$ along any particular line of force of $\boldsymbol{B}_t$ depends on the variation of the radius of curvature of *all* the lines u = const which are enclosed by the particular field line we are considering. For the general case, equation (73) implies that the radius of curvature will be larger in regions with higher azimuthal field intensity. This can be clearly seen in the simple equilibrium plotted in Figure 4. In the places in which the tube is compressed from outside, the field lines are closer together because of the boundary condition (72) and the curvature is smaller: the weaker "radial" gradient of the total pressure excess (Fig. 4*f*) allows the inward magnetic tension to diminish. On the contrary, in the places in which the tube is expanded, F and R_c must become smaller.

6. EFFECT OF GRAVITY AND EXTERNAL PRESSURE FLUCTUATIONS: VERTICAL MOTION OF A TWISTED HORIZONTAL FLUX TUBE

In §§ 2 and 3 we have studied the effect of gravity on the equilibrium of a horizontal magnetic flux tube. In § 5 we have considered the perturbation resulting from an arbitrary fluctuation of the external pressure distribution along the boundary. It is natural to ask now if an equilibrium is possible when the tube *feels* the effects of both gravity and the external pressure fluctuations. As explained in the discussion, § 7, a solution to this problem could enable us to understand the transverse structure of a horizontal magnetic flux tube rising in the convection zone of the Sun.

In order to make use of the results obtained in previous sections, we separate the external pressure into $p_e[z(u, \phi)]$, which is determined by the stratification of the polytropic background medium, and $p_{ef}(\phi)$, which represents the perturbation of p_e along the boundary of the tube. Finally, we define the pressure and density excess as in § 2, equation (18), even though now p_e no longer corresponds to the whole external pressure. As can be seen by performing the integration of the momentum equation (2) over the whole tube section $\mathscr{D}$,

$$0 = -\int_{\partial\mathscr{D}} p_{ef}\,\boldsymbol{n}\,dl_\phi + \boldsymbol{G}\iint_{\mathscr{D}}\Delta\rho\,d\Sigma, \tag{74}$$

the net force exerted by $p_{ef}(\phi)$ along the boundary must compensate for the buoyancy force of the tube.

In the coordinate system instantaneously at rest with respect to the tube, the resulting equilibrium equations coincide with those of § 2, equations (22)–(25). The only difference appears in the boundary condition: the total pressure excess at the boundary is now different from zero, and we have

$$\frac{\partial p_{ef}}{\partial \phi} = \frac{\partial \Delta p}{\partial \phi} + \frac{b^2}{8\pi}\frac{\partial F}{\partial \phi} \quad \text{at } u = U, \tag{75}$$

instead of equation (24). Equation (75) shows that the shape of the tube results from the concurrent action of gravity (via Δp) and of the external pressure fluctuation along the boundary. The mutual separation of the field lines of $\boldsymbol{B}_t$ close to the

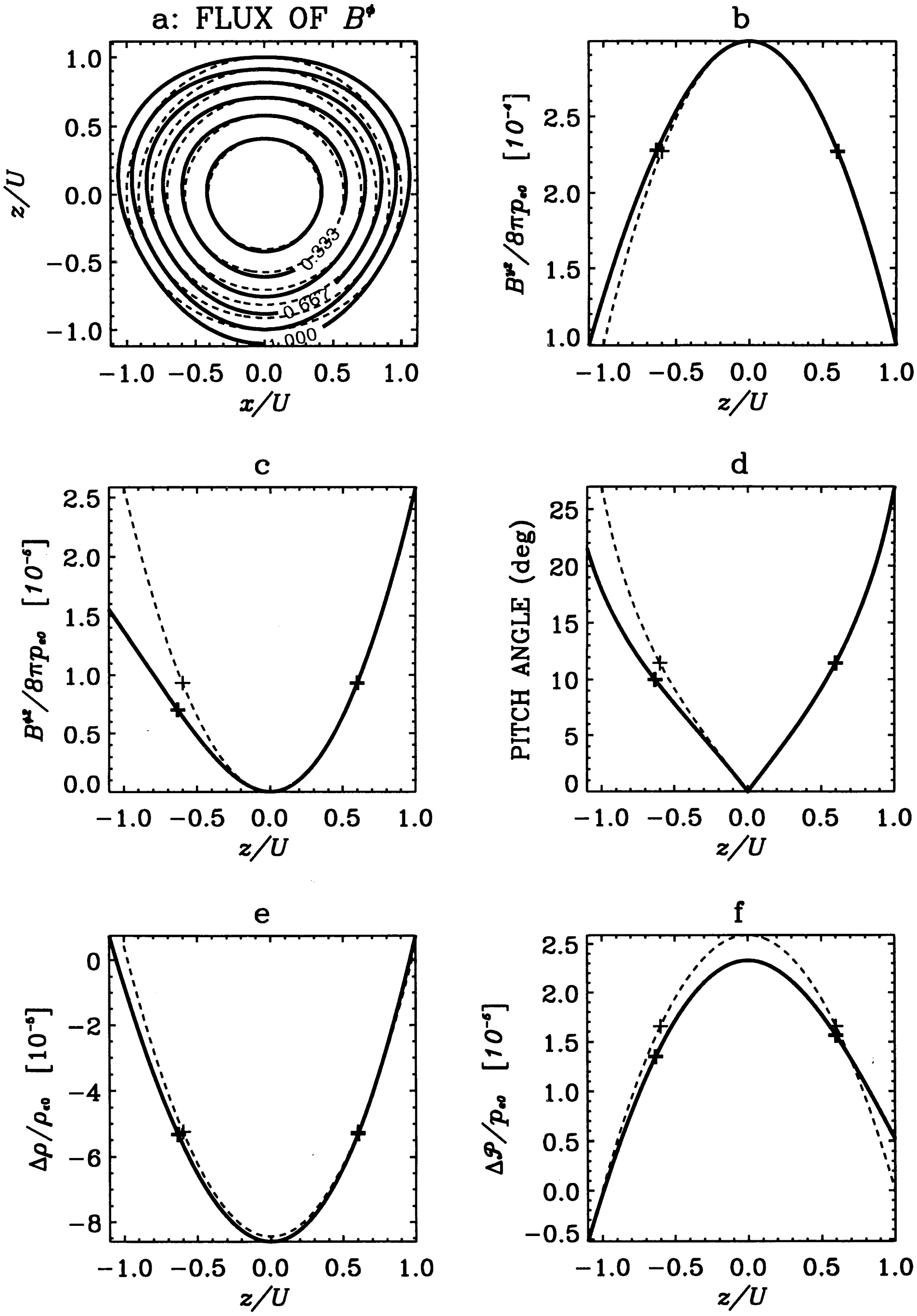

FIG. 4.—Twisted magnetic flux tube embedded in a constant pressure atmosphere and perturbed along its boundary by an external pressure fluctuation of the form $p_{ef} = b_0^2/8\pi P_{c3} \cos(3\phi)$. The algebraic form of the functions plotted here are given in Appendix C ($\alpha_\phi = 3 \times 10^{-5}$, $\alpha_y = 2 \times 10^{-4}$, $\xi = -2 \times 10^{-5}$, $m_\phi = 1$, and $m_y = 1$).

boundary will therefore reflect both effects. This can be seen in the following equation, which is a generalization of equation (26) of § 2:

$$\frac{\partial F}{\partial \phi} = \frac{8\pi p_e}{b^2}\frac{\Delta\rho}{\rho_e}\frac{U}{H_e}\frac{\partial}{\partial\phi}\left(\frac{z}{U}\right) + \frac{\partial}{\partial\phi}\left(\frac{8\pi p_{ef}}{b^2}\right). \quad (76)$$

Thus, if the deformation of the tube shape is to be small, *both* terms on the right-hand side of this equation must be much smaller than one (see also the discussion in § 7).

Assume, as before, that the effect of gravity (as measured by ϵ) and of the pressure fluctuations along the boundary is small. The pressure fluctuation p_{ef} can be split up into two terms, each one having a different effect on the tube: a drag contribution, whose integration along the boundary gives the downward force necessary to cancel the buoyancy of the tube; and a second term which perturbs the internal plasma without producing a net force on the tube. Both effects need not be of the same order of magnitude: the effect of the drag force has to be of order ϵ if it is to compensate the total buoyancy force, but the rest of the perturbation caused by the pressure fluctuations may be of different order. For simplicity, however, we will assume in the following that *all* perturbations appear in the momentum equations and boundary conditions as terms of order ϵ with respect to the zero-order equilibrium. A generalization of this constraint is not difficult to obtain but is not of primary interest in this paper.

We define zero-order and first-order symbols as before. The projection of equation (74) onto the z- and x-direction yields, up to first order,

$$0 = -\int_0^{2\pi} p_{\rm ef}\, U \cos\phi\, d\phi - 2\pi G \int_0^U \Delta\rho_0\, u\, du\,, \qquad (77)$$

$$0 = -\int_0^{2\pi} p_{\rm ef}\, U \sin\phi\, d\phi\,, \qquad (78)$$

respectively. Equation (78) implies that the pressure fluctuation cannot contain any term of the form sin ϕ, otherwise a net lateral force on the tube would ensue which is not compensated by any other force. Thus, the most general form for $p_{\rm ef}$ is given by

$$p_{\rm ef}(\phi) = \frac{b_0^2}{8\pi}(U)\, P(\phi) + p_{e0}\, P_{\rm drag} \cos\phi\,, \qquad (79)$$

where $P(\phi)$ is defined in equation (64). Introducing equation (79) into equation (77), one then obtains

$$P_{\rm drag} = -\epsilon\, \frac{\overline{\Delta\rho_0}}{\rho_{e0}}(u = U)\,, \qquad (80)$$

ensuring that the drag force effectively compensates the total buoyancy force.

To first order, the boundary condition in equation (75) yields

$$\frac{b_0^2}{8\pi}\frac{\partial P}{\partial\phi} - p_{e0}\, P_{\rm drag} \sin\phi = \Delta p_0 \frac{\partial \Delta p_1}{\partial\phi} - \frac{b_0^2}{4\pi}\frac{\partial}{\partial u}(u r_{1,\phi})\,. \qquad (81)$$

The rest of the zero- and first-order equations which describe the equilibrium are the same as in § 3 (eqs. [32]–[37] with eq. [81] instead of eq. [36]).

The zeroth-order equilibrium is determined as in § 3.2 with the only difference that now $\Delta\rho_0$ need not satisfy the no-buoyancy condition of equation (38). Next, we must find the solution $r_1(u, \phi)$ to the ϕ derivative of the "radial" momentum equation, equation (34), together with the boundary conditions in equations (81) and (37). The r_1 we are searching for is simply the sum of the solutions obtained in §§ 3 and 5: if we introduce

$$r_1 = a_0(u) + a_1(u)\cos\phi + \sum_{k=2}^{\infty} [a_{ck}(u)\cos(k\phi) + a_{sk}(u)\sin(k\phi)] \qquad (82)$$

into the momentum equation (34), we obtain that a_1 is given by equation (44) of § 3 and that the a_{ck} and a_{sk} with $k \geq 2$ must be a solution of equation (71) of § 5. The difference with the previous sections arises from the boundary condition in equation (81) which, for a_1, gives

$$-p_{e0}\, P_{\rm drag} = \Delta\rho_0\, GU + \frac{b_0^2}{4\pi}\frac{\partial}{\partial u}(u a_1) \quad \text{at } u = U\,, \qquad (83)$$

whereas for $k \geq 2$ we obtain the boundary condition of equation (68) like in § 5. Once r_1 is known, the other first-order variables are obtained in the same way as in § 3.2.

As in §§ 3 and 5, we have calculated in Appendix D the particular equilibrium resulting when b_0, b_1, B_0^y, and B_1^y have the power-law form given in Appendix B. In Figure 5 we show the case for which $b_1 = B_1^y = 0$ and where the external pressure fluctuation is $p_{\rm ef} = p_{e0}\, P_{\rm drag} \cos\phi + b_0^2/8\pi P_2 \cos(2\phi)$. In those plots, the signature of both perturbations is visible. As in Figure 2, the core of the tube is more buoyant than the outer layers. This *differential buoyancy* shifts upward the $\boldsymbol{B}_t$-field lines. The potential flow–like term cos (2ϕ) of the external perturbation, in turn, forces the transverse magnetic field lines to bend (see § 5). Finally, the cos ϕ term of $p_{\rm ef}$ exerts a net downward drag force which counteracts the total buoyancy of the tube.

7. DISCUSSION

In this paper we have studied the equilibrium resulting from the perturbation of an axisymmetric flux tube through external agents. We have included a few cases relevant for the study of flux tubes in the solar convection zone: first, the perturbation caused by the gravitational field (including the effects of the ensuing external stratification); second, the perturbation caused by an external flow field via a nonuniform pressure distribution on the tube surface and, finally, a combination of the two. We have written the general equilibrium equations using a system of flux coordinates (§ 2). The equations obtained correspond to the general case of arbitrary plasma β with gravity, i.e., generally to non–force-free fields. They are complex, so that, with a view to applications in the convection zone and perhaps in the corona, we have considered the case of a *weak* external stratification (i.e., small change of the external variables across the tube radius) and small pressure fluctuations on the tube boundary. We have found solutions to the equations for generalized power-law distributions of the unperturbed and perturbed physical variables in the tube. In the following we discuss several points concerning the applications of the present study, the information on the transverse structure it provides, and some stability considerations.

7.1. *Tubes Rising in the Convection Zone*

A first application of the foregoing equations is to flux tubes (specially in the form of toroidal flux *rings*) rising through the convection zone toward the surface. Although the rise of tubes in general is a dynamical phenomenon, it customarily proceeds in a quasi-static manner. One condition for this to happen is that the tube adopt at every instant the terminal velocity, $v_{\rm term}$, in which the drag force exerted by the flow surrounding the tube exactly compensates the forces that drive the rise (in our case, the total buoyancy force on the tube). If we assume the flow resistance to be given by the aerodynamic drag formula (e.g., Batchelor 1967), then the drag force is proportional to $v_\perp^2/U$, the square of the relative speed of the tube as a whole with respect to the surrounding divided by the tube radius. If so, the condition for instantaneous adjustment of rise speed and terminal velocity is that the tube radius be small compared to the length scale for spatial variation of the external variables (Moreno-Insertis 1983), e.g., the pressure scale height. Hence, it coincides with the condition of a weak external stratification. Moreover, $v_{\rm rise} \approx v_{\rm term}$ implies that

$$\frac{v_{\rm A}^2}{v_{\rm rise}^2} \gtrsim \mathcal{O}\left(\frac{H_e}{U}\right), \qquad (84)$$

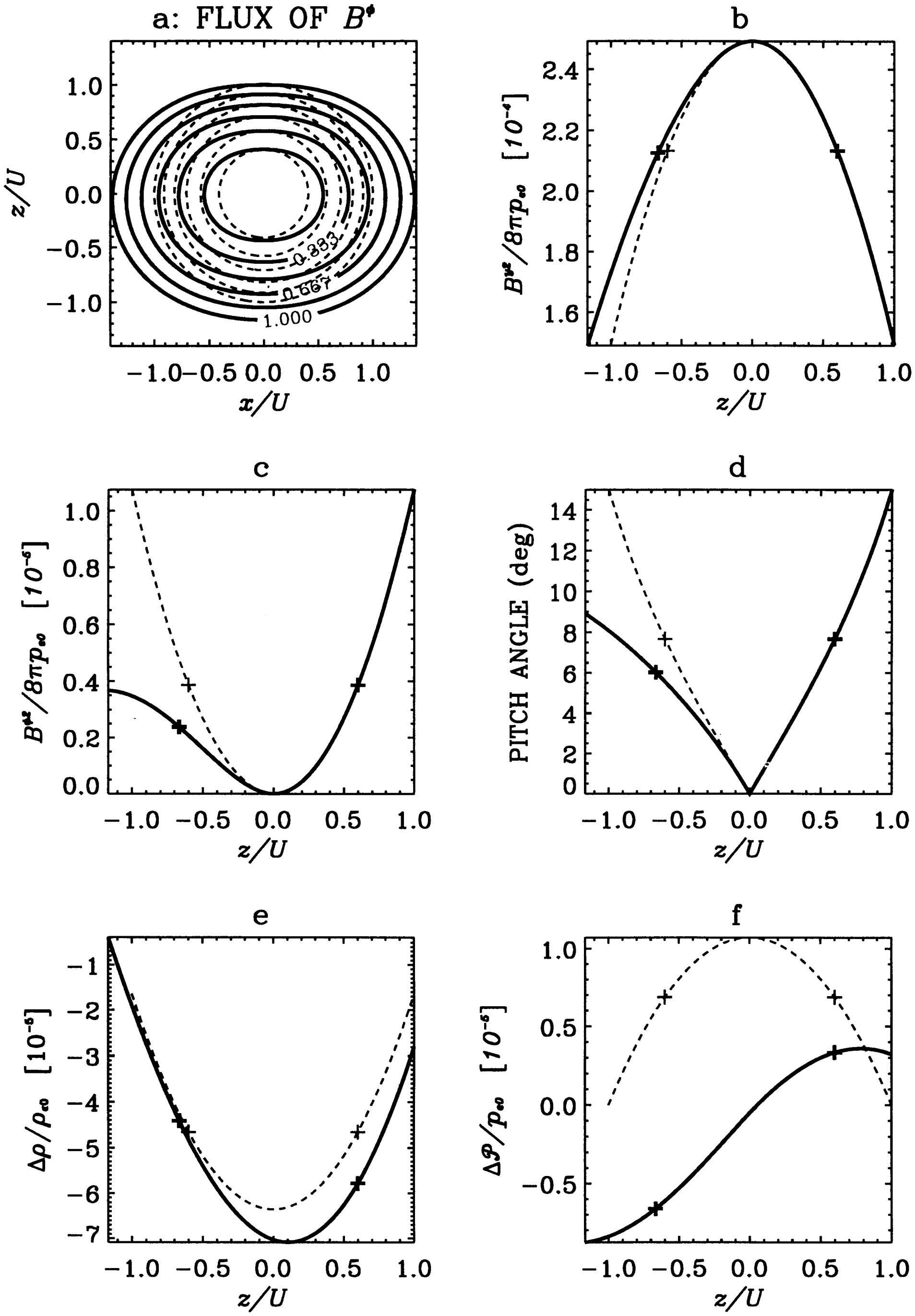

FIG. 5.—Twisted horizontal magnetic flux tube embedded in a stratified polytropic atmosphere which simulates the convection zone of the Sun at a depth of 50,000 km. The mean density excess of the tube is negative, and the resulting buoyancy force is canceled by the net downward force exerted by the external pressure fluctuation along the boundary. Apart from the contribution to the drag, p_{ef} produces also a potential-flow–like perturbation proportional to $\cos(2\phi)$ along the boundary. The exact form of the functions plotted here is given in Appendix D. The parameters are $\alpha_\phi = 10^{-5}$, $\alpha_y = 2 \times 10^{-4}$, $\xi = -10^{-4}$, and $m_\phi = m_y = 1$.

so that, if $U \ll H_e$, then the timescale for the establishment of the magnetic equilibrium in the tube is much smaller than the timescale of change of properties in the medium surrounding the tube:

$$\frac{U}{v_A} \ll \frac{H_e}{v_{\rm rise}} . \tag{85}$$

In deriving equation (84), $\Delta\rho/\rho$ has been assumed to be of order the magnetic buoyancy. Summarizing, whenever $U/H_e \ll 1$, the rise may be said to be quasi-static: the inertial terms are negligible in the equations, and the tube adopts at each height the structure of a static tube that were subject to the same local external stratification and pressure perturbations at the boundary as our rising tube. The condition of the smallness of the tube radius is amply satisfied in the convection zone for the field strength and magnetic flux values expected in tubes that later give rise to active regions (Moreno-Insertis 1992; Caligari, Moreno-Insertis, & Schüssler 1995). The approximation of quasi-static rise will only break down in the uppermost scale-heights of the convection zone: in addition to the large buoyancy of the tube associated with the entropy difference with the surroundings, the external scale height becomes small and the tube radius very large because of the expansion associated with the rise.

Considering the pressure fluctuations around the boundary of the rising tube, an additional condition has to be met for the perturbation scheme to be applicable. We saw in § 2 that the longitudinal field is constant on the azimuthal field lines and that the internal gas pressure is hydrostatically stratified along them. This remains true when the external flow perturbation is added (see § 5). Thus, it is the spatial variations in the azimuthal field which must compensate for the external pressure fluctuations along the boundary. As a result, if the deformation is to be small, one has to require that

$$p_{\rm ef} \ll \frac{b^2}{8\pi} . \qquad (86)$$

Now, $p_{\rm ef}$ will be of the order of the ram pressure $\rho v_\perp^2$, so that the condition for the azimuthal magnetic field to absorb the external flow pressure without large deformation is that the rise speed be much smaller than the Alfvén speed of the azimuthal field, which is more restrictive than equation (85). Setting again $v_{\rm rise} \approx v_{\rm term}$ and the buoyancy of the tube of order the magnetic buoyancy, one can then write the foregoing condition in equation (86) in the following approximate form:

$$\frac{U}{H_e} \ll \frac{b^2}{B^{y2}} . \qquad (87)$$

These conditions can also be inferred directly from equation (76) if one requires that the deformation of the tube from the axisymmetric shape be small. The inequalities (87) and (86) provide a lower bound (to order of magnitude) for the azimuthal fields that can easily resist the external deformation. The actual ability of the azimuthal field to absorb the external pressure fluctuations will change along the rise: both ratios on the right- and left-hand side of equation (87) typically grow as the flux tube rises through the convection zone, as a consequence of the general increase of the tube radius (Parker 1979). In the next paper of this series, a study of the processes taking place in rising tubes will be presented, and the actual dependence of both sides of equation (87) on the level reached by the tube in the convection zone will be studied in detail.

7.2. *The Transverse Structure of the Tube and the Thin Flux-Tube Approximation*

The equations developed in this paper allow one to study in detail the effects on the magnetic structure of (1) the *differential buoyancy* of the various parts of the tube cross section and (2) an arbitrary external pressure distribution on the tube surface. The main effect of the gravity is to shift vertically the centers of the azimuthal field lines in the interior of the tube without changing their circular shape, so that the resulting changes in magnetic pressure and tension exactly counteract the differences in buoyancy in the tube cross section (§§ 2 and 3). On the other hand, an external pressure distribution on the boundary with Fourier components of circular wavenumber greater than 1 forces the field lines to bend to achieve the equilibrium (§ 5). If, in addition, the external pressure fluctuation contains a cosine and/or a sine term, it will produce nonvanishing resulting forces on the tube ("drag" and "lift," respectively).

In the case of our prototype rising flux rope, both effects of gravity and external flow field determine the shape of the tube (§ 6 and Fig. 5): the external flow bends the azimuthal field lines, and in the interior the gravity shifts the centers of the field lines without deforming them. The equilibrium is achieved because the drag exactly counteracts the total buoyancy of the tube. For a tube of this sort, the density deficit typically has its maximum value in the center and diminishes toward the boundary. Thus, in general, we expect the azimuthal field of a rising magnetic flux tube to be stronger in the upper half than in the lower half of one of its sections. Keeping in mind that the stabilizing agent against the external flow is the azimuthal field, we deduce that the stability of the tube will be enhanced in its frontal part and diminished at its rear (see also § 7.3).

There are important differences between the perturbation scheme introduced in this paper and the customary expansion procedure known in the literature as the thin flux-tube approximation. As pointed out in § 4, the latter does not allow for a large twist of the tube (i.e., the pitch angle must be small), so its applicability to rising tubes in the convection zone (as well as possibly to flux tubes in the corona) is limited. In fact, by going close enough to the axis one encounters a "thin flux-tube region" within every more general tube, since the pitch angle tends to zero toward the axis. However, that region is arbitrarily small in strongly twisted tubes; in rising tubes, in particular, the fraction of the tube occupied by this region may become very small. In addition, the pressure and density excess fluctuations may be larger in the tubes treated in this paper than in *slender* flux tubes, so that even the core of the tubes may not be easy to study with the thin flux-tube approach.

7.3. *Stability Considerations*

The stability of the basis (or zero-order) axisymmetric equilibrium has been studied in the literature, in particular for the case of force-free magnetic configurations (Anzer 1968; Parker 1979; Priest 1982). The first of those papers, in particular, showed that a cylindrically symmetric, twisted force-free tube is unconditionally unstable, although there may exist configurations close to the original one but with helical symmetry which are stable. In the non–force-free case of the present paper, the presence of gas pressure and gravity complicates the stability study considerably. Giachetti, Van Hoven, & Chiuderi (1977) have shown in a series of papers that the stability of a coronal flux tube is enhanced by a positive gas pressure gradient in the radial direction. The range of plasma β in their papers, however, is of order unity, and the extrapolation of their results to our high-β problem is not straightforward. Gravity, on the other hand, brings in a range of further possible instabilities: for instance, Rayleigh-Taylor and convective stability criteria will mingle with purely MHD conditions, and this applies also to the weakly stratified case. In addition, instabilities with flows along field lines ("Parker"-like instabilities) for twisted flux ropes may appear, rendering the global stability problem quite intractable. Finally, for the rising tubes, motion-related instabilities should be included (e.g., Kelvin-Helmholtz).

A few comments on partial aspects of the stability problem can nevertheless be of interest at this point. The inequalities

(86) and (87) are conditions for the external flow field to cause only small perturbations to the axisymmetric equilibrium. These conditions are similar in form to the stability conditions obtained by Tsinganos (1980) (with, in his case, $\lesssim$ signs in place of our $\ll$ signs). The latter author has studied the *hydrodynamic* instability of buoyant fields. Making an analogy between a nonmagnetic fluid cylinder and a magnetic flux rope, he found that a magnetic flux tube embedded in a fluid velocity field is hydrodynamically unstable if it has no azimuthal component. Thus, all this indicates that a net amount of twist may be important for the flux tube to reach the surface of the Sun in a coherent state.

The amount of twist cannot be indefinitely large because the presence of an azimuthal component renders the tube vulnerable to the kink (lateral, helical) instability (e.g., Priest 1982). As a minimum condition, the intensity of the azimuthal field is limited by the requirement that the net tension in the tube above background be positive (see Parker 1979). That is, for an axisymmetric flux tube, the magnetic pressure of B^ϕ must not exceed the magnetic tension of B^y:

$$2 > \frac{\overline{B^{\phi 2}}}{\overline{\overline{B^{y2}}}} . \tag{88}$$

If the total tension becomes negative, the tube is placed under longitudinal compression and therefore slips into helical form.

7.4. *Tubes with Curved Longitudinal Field and Further Applications*

The configurations of this paper may help to understand possible flux-tube equilibria in the convection zone: these include tubes in the overshoot region (Moreno-Insertis, Schüssler, & Ferriz-Mas 1992; Ferriz-Mas & Schüssler 1994), tube equilibria resulting from a meridional flow (van Ballegooijen & Choudhuri 1988), possible nonlinear equilibria resulting from the evolution of Parker-unstable tubes with a strong initial magnetic field (Anton 1983), as well as the quasi-equilibrium of rising flux rings discussed in § 7.1.

In all the above-mentioned equilibria, the longitudinal structure of the tube, i.e., the possible variations of its physical and geometric variables along the axis, can be ignored, as has been done in the present paper: one can assume that the influence of the longitudinal structure on the equilibrium depends on the ratio, η, of the inner radius of the tube cross section to the radius of curvature of the torus. This ratio is smaller than the parameter ϵ which controls the importance of the gravity perturbation by the same factor as the ratio of the local scale-height to the solar radius, i.e., typically a factor 10 smaller. In fact,

$$\eta \approx 4 \times 10^{-3} \frac{\Phi_{22}}{B_5} , \tag{89}$$

with Φ_{22} the magnetic flux in units of 10^{22} Mx and B_5 the magnetic field in units of 10^5 G. Even if η became comparable to ϵ, one could study the effect of the curvature of the flux ring without great difficulty; in fact, it produces a deformation similar to the effect of differential buoyancy (Emslie & Wilkinson 1994). The longitudinal structure of the tube may become dominant in determining the equilibrium whenever the radius of curvature of the tube becomes smaller than the local pressure scale height. Hence, the results of the present study may not be directly applicable to coronal tubes nor to the very late stages of the rise of *sea-serpent*–shaped tubes (also called Ω-loops) in the convection zone, shortly before they arrive at the photosphere.

The authors are grateful to A. W. Hood for many interesting discussions and suggestions on this manuscript. Interesting discussions with E. R. Priest and B. Roberts are also acknowledged. This work was partially funded through the DGICYT project C91-0530.

APPENDIX A

GEOMETRY

The natural basis in the coordinates (u, ϕ, y) is (see Auslander & Mckenzie 1977; Kobayashi & Nomizu 1963)

$$\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial \phi}, \frac{\partial}{\partial y}\right), \tag{A1}$$

and an arbitrary vector $\boldsymbol{v}$ can be written in this basis as

$$\boldsymbol{v} = \tilde{v}^u \frac{\partial}{\partial u} + \tilde{v}^\phi \frac{\partial}{\partial \phi} + \tilde{v}^y \frac{\partial}{\partial y} , \tag{A2}$$

so that $\tilde{v}^u$, $\tilde{v}^\phi$, and $\tilde{v}^y$ are the contravariant components of $\boldsymbol{v}$ in the basis (A1). The matrix g of the metric is

$$g = \begin{pmatrix} g_{uu} & g_{u\phi} & g_{uy} \\ g_{\phi u} & g_{\phi\phi} & g_{\phi y} \\ g_{yu} & g_{y\phi} & g_{yy} \end{pmatrix} = \begin{pmatrix} r_{,u}{}^2 & r_{,u} r_{,\phi} & 0 \\ r_{,u} r_{,\phi} & r_{,\phi}{}^2 + r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{A3}$$

Here g is not diagonal because the basis is not orthogonal. The gradient of an arbitrary scalar function f is the contravariant vector corresponding to the one-form df, i.e.,

$$\nabla f = \left(g^{uu} \frac{\partial f}{\partial u} + g^{\phi u} \frac{\partial f}{\partial \phi}\right) \frac{\partial}{\partial u} + \left(g^{u\phi} \frac{\partial f}{\partial u} + g^{\phi\phi} \frac{\partial f}{\partial \phi}\right) \frac{\partial}{\partial \phi} + \frac{\partial f}{\partial y} \frac{\partial}{\partial y} , \tag{A4}$$

and the divergence is

$$\nabla \cdot \boldsymbol{v} = \left\{ \frac{\partial}{\partial u}\left[\sqrt{\det(g)}\,\tilde{v}^u\right] + \frac{\partial}{\partial \phi}\left[\sqrt{\det(g)}\,\tilde{v}^\phi\right] + \frac{\partial}{\partial y}\left[\sqrt{\det(g)}\,\tilde{v}^y\right]\right\} \frac{1}{\sqrt{\det(g)}} . \tag{A5}$$

Here $g^{u\phi}$ is an element of the inverse matrix g^{-1} of g. For the covariant differentiation $(\boldsymbol{B}_t \cdot \nabla)\boldsymbol{B}_t$ of the transverse magnetic field

$$\boldsymbol{B}_t = \tilde{B}^\phi \frac{\partial}{\partial \phi} , \tag{A6}$$

we obtain

$$(\boldsymbol{B}_t \cdot \nabla)\boldsymbol{B}_t = \frac{(\tilde{B}^\phi)^2}{2}\left[g^{uu}\left(2\frac{\partial g_{\phi u}}{\partial \phi} - \frac{\partial g_{\phi\phi}}{\partial u}\right) + g^{u\phi}\frac{\partial g_{\phi\phi}}{\partial \phi}\right]\frac{\partial}{\partial u} + \left\{\frac{(\tilde{B}^\phi)^2}{2}\left[g^{\phi u}\left(2\frac{\partial g_{\phi u}}{\partial \phi} - \frac{\partial g_{\phi\phi}}{\partial u}\right) + g^{\phi\phi}\frac{\partial g_{\phi\phi}}{\partial \phi}\right] + \frac{\partial}{\partial \phi}\left[\frac{(\tilde{B}^\phi)^2}{2}\right]\right\}\frac{\partial}{\partial \phi} . \tag{A7}$$

The corresponding basis of unit vectors is

$$(\boldsymbol{e}_u, \boldsymbol{e}_\phi, \boldsymbol{e}_y) = \left(\frac{1}{\sqrt{g_{uu}}}\frac{\partial}{\partial u}, \frac{1}{\sqrt{g_{\phi\phi}}}\frac{\partial}{\partial \phi}, \frac{\partial}{\partial y}\right) , \tag{A8}$$

and we shall write the coefficients of a vector $\boldsymbol{v}$ in this basis without tilde. This yields the following relation between the coefficients of the natural and unit basis:

$$v^u = \tilde{v}^u \sqrt{g_{uu}} , \qquad v^\phi = \tilde{v}^\phi \sqrt{g_{\phi\phi}} , \qquad v^y = \tilde{v}^y . \tag{A9}$$

Using equations (A5) and (A6), equation (1) can be transformed into

$$\frac{\partial}{\partial \phi}\left[\sqrt{\det(g)}\,\tilde{B}^\phi\right] = 0 . \tag{A10}$$

Taking then the scalar product of equation (2) with $\partial/\partial u$ and $\partial/\partial \phi$ and of equation (3) with $\partial/\partial y$, we obtain

$$0 = -\frac{\partial}{\partial u}\left[p + \frac{B^{y2}}{8\pi} + \frac{(\tilde{B}^\phi)^2 g_{\phi\phi}}{8\pi}\right] - \rho G\frac{\partial z}{\partial u} + \frac{(\tilde{B}^\phi)^2}{8\pi}\left(2\frac{\partial g_{\phi u}}{\partial \phi} - \frac{\partial g_{\phi\phi}}{\partial u}\right) + g_{\phi u}\frac{\partial}{\partial \phi}\left[\frac{(\tilde{B}^\phi)^2}{8\pi}\right] , \tag{A11}$$

$$0 = -\frac{\partial p}{\partial \phi} - \rho G\frac{\partial z}{\partial \phi} , \tag{A12}$$

$$0 = \frac{\partial B^y}{\partial \phi} , \tag{A13}$$

respectively. The boundary condition yields

$$p_e = p + \frac{B^{y2}}{8\pi} + \frac{(\tilde{B}^\phi)^2 g_{\phi\phi}}{8\pi} \qquad \text{at } u = U . \tag{A14}$$

Introducing equations (A3) and (A9) in the system of equations (A10)–(A14), we obtain easily equations (10)–(16).

APPENDIX B

POWER-LAW MODEL FOR A HORIZONTAL MAGNETIC FLUX TUBE IN EQUILIBRIUM WITH A STRATIFIED EXTERNAL MEDIUM

To illustrate the study carried out in § 3, we present here a simple power-law solution to the system of linearized equations (30)–(38). Before dealing with the equilibrium in the interior of the tube, we lay down the stratification of the external medium. Within the small-radius approximation, one can use a polytropic expression for the external atmosphere as follows:

$$p_{e1} = -\epsilon\left(\frac{u}{U}\right)\cos\phi , \qquad \rho_{e1} = \epsilon(1-\nabla)\left(\frac{u}{U}\right)\cos\phi , \qquad H_{e1} = -\epsilon\nabla\left(\frac{u}{U}\right)\cos\phi , \qquad s_{e1} = -\epsilon\frac{\delta c_p}{s_{e0}}\left(\frac{u}{U}\right)\cos\phi . \tag{B1}$$

In the foregoing, ∇ is the logarithmic temperature gradient, $\nabla = d\log T_e/d\log p_e$, and δ is the *superadiabaticity* of the external stratification, $\delta = \nabla - \nabla_{ad}$.

The azimuthal and longitudinal magnetic field components can be chosen freely. For the zero-order equilibrium, we take here the following power laws:

$$\frac{B_0^{\phi 2}}{8\pi p_{e0}} = \frac{b_0^2}{8\pi p_{e0}} \stackrel{\text{def}}{=} \alpha_\phi\left(\frac{u}{U}\right)^{2m_\phi} , \tag{B2}$$

$$\frac{B_0^{y2}}{8\pi p_{e0}} \stackrel{\text{def}}{=} \alpha_y + \xi\left[\left(\frac{u}{U}\right)^{2m_y} - \frac{1}{m_y+1}\right] , \tag{B3}$$

where α_ϕ, α_y, ξ, m_ϕ, and m_y are constant parameters that fix the limit values of the field at the center and boundary of the tube and the steepness of the profile. Those parameters can be chosen freely with the restriction that the functions on the right-hand side be finite and positive and that $m_\phi \geq 1$ to avoid an infinite current density along the axis. From equation (40), there follows the total and gas pressure profiles; e.g., for the total pressure,

$$\frac{\Delta \mathscr{P}_0}{p_{e0}} = \frac{\alpha_\phi}{m_\phi}\left[1 - \left(\frac{u}{U}\right)^{2m_\phi}\right]. \tag{B4}$$

We are assuming for simplicity that the internal entropy s_0 is spatially constant (see § 3.2.3), so that the third functional degree of freedom of the zero-order equilibrium, $\Delta\rho_0$, is given as a function of Δp_0 through

$$\frac{\Delta\rho}{\rho_e} = e^{-\Delta s/c_p}\left(1 + \frac{\Delta p}{p_e}\right)^{1/\gamma} - 1 \approx -\frac{\Delta s}{c_p} + \frac{1}{\gamma}\frac{\Delta p}{p_e}. \tag{B5}$$

The approximate expression on the right is only valid for large enough plasma beta and small enough entropy difference between inside and outside the tube [which are very good approximations for flux tubes in the convection zone except its uppermost levels; e.g., for $B = 10^5$ G, $\beta = \mathscr{O}(10^5)$ at the bottom of the convection zone]. The nonbuoyancy integral condition in equation (38), finally, fixes the value of the constant internal entropy:

$$0 = -\frac{\Delta s_0}{c_p} - \frac{\alpha_y}{\gamma} = \frac{\overline{\Delta\rho_0}}{\rho_{e0}}(U). \tag{B6}$$

Once the zero-order values are known, r_1, which determines the geometry of the tube, follows from equations (43) and (44), viz.,

$$a(u) = \epsilon\left[\frac{1}{4\gamma} + \frac{\xi}{4\gamma\alpha_\phi}K\left(\frac{u}{U}\right)^{2m_y - 2m_\phi}\right]\left(\frac{u}{U}\right) \quad \text{with } K \stackrel{\text{def}}{=} \frac{m_y}{(m_y+1)(m_y - m_\phi + 1)}. \tag{B7}$$

To avoid singularities in the center, we require that

$$2m_y - 2m_\phi + 1 \geq 0. \tag{B8}$$

Condition (B8) is a restriction to the functional degrees of freedom of the zero-order equilibrium (as seen in § 3.2, after formula [44]). It has a general significance for flux tubes with arbitrary (i.e., not necessarily power-law) profiles. For, close to the axis, one can always expand b_0 and B_0^y in Frobenius series

$$b_0 = b_{0m_\phi}u^{m_\phi} + b_{0m_\phi+1}u^{m_\phi+1} + \cdots, \qquad B_0^y = B_{0m_y}^y u^{m_y} + B_{0m_y+1}^y u^{m_y+1} + \cdots, \tag{B9}$$

where, as above, $m_\phi \geq 1$. The corresponding power expansion of equation (44) is then

$$a = c_1 u + \mathscr{O}(u^2) + c_2 u^{2m_y - 2m_\phi + 1} + \mathscr{O}(u^{2m_y - 2m_\phi + 2}), \tag{B10}$$

with c_1 and c_2 positive constants. The nonsingularity of equation (B10) at the axis is then guaranteed by equation (B8).

As before, we define power series for B_1^ϕ and B_1^y through

$$\frac{2b_0^2 b_1}{8\pi p_{e0}} \stackrel{\text{def}}{=} \epsilon\left[\alpha'_\phi\left(\frac{u}{U}\right)^{m'_\phi}\right], \tag{B11}$$

$$\frac{2B_0^{y2}B_1^y}{8\pi p_{e0}} \stackrel{\text{def}}{=} \epsilon\left\{\alpha'_y + \xi'\left[\left(\frac{u}{U}\right)^{m'_y} - \frac{2}{m'_y + 2}\right]\right\}, \tag{B12}$$

where, again, α'_ϕ, α'_y, ξ', m'_ϕ, and m'_y are constant parameters. The corresponding solution for the total pressure and entropy are

$$\frac{\Delta\mathscr{P}_0 \Delta\mathscr{P}_1^*}{p_{e0}} = \epsilon\left\{\frac{2\alpha'_\phi}{m'_\phi}\left[1 - \left(\frac{u}{U}\right)^{m'_\phi}\right] + C_1\left[1 - \left(\frac{u}{U}\right)\right] + C_2\left[1 - \left(\frac{u}{U}\right)^{2m_\phi+1}\right] + C_3\left[1 - \left(\frac{u}{U}\right)^{2m_y+1}\right]\right\}, \tag{B13}$$

$$\frac{\Delta s_0 \Delta s_1}{c_p} = \frac{s_0 s_1}{c_p} + \epsilon\,\delta\left(\frac{u}{U}\right)\cos\phi. \tag{B14}$$

The pressure distribution can be easily derived from equations (B13), (B11), and (B12). The constants in equation (B13) are given by

$$C_1 = \frac{1}{\gamma}\left(\frac{\alpha_\phi}{m_\phi} - \alpha_y + \frac{\xi}{m_y+1}\right) - \frac{\Delta s_0}{c_p}, \qquad C_2 = \frac{1}{\gamma}\left[\frac{-\alpha_\phi(m_\phi+2)}{2m_\phi(2m_\phi+1)}\right], \qquad C_3 = \frac{1}{\gamma}\left[\frac{\xi(K-2)}{2(2m_y+1)}\right]. \tag{B15}$$

APPENDIX C

POWER-LAW MODEL FOR A MAGNETIC FLUX TUBE PERTURBED BY AN ARBITRARY EXTERNAL PRESSURE DISTRIBUTION

In this Appendix we calculate a particular solution to the zero- and first-order systems of equations (32)–(33) and (66)–(69), which describe the equilibrium of a magnetic flux tube perturbed by an arbitrary external pressure distribution of the form of equation (64).

For simplicity we will use the same zero-order magnetic field distributions as in Appendix B, namely, equations (B2)–(B4). The first-order equilibrium will depend on the external pressure fluctuations. If we assume that

$$P = P_0 + \sum_{k=k_0}^{\infty} [P_{ck} \cos (k\phi) + P_{sk} \sin (k\phi)] , \tag{C1}$$

where k_0 is an integer greater than or equal to 2, then the integration of equation (71), with b_0 given by equation (B2), leads to the following solution to equations (66), (68), and (69):

$$r_1 = a_0 + \sum_{k=k_0}^{\infty} [a_{ck}(u) \cos (k\phi) + a_{sk}(u) \sin (k\phi)] , \tag{C2}$$

with

$$a_0(u) = - \sum_{k=k_0}^{\infty} a_{ck}(u) , \tag{C3}$$

$$a_{\binom{ck}{sk}}(u) = A_{\binom{ck}{sk}}\left(\frac{u}{U}\right)^{-1-m_\phi+\sqrt{-1+k^2+m_\phi^2}} , \tag{C4}$$

$$A_{\binom{ck}{sk}} = \frac{-P_{\binom{ck}{sk}}}{2\left(-m_\phi + \sqrt{-1+k^2+m_\phi^2}\right)} . \tag{C5}$$

Here a_0 follows from the boundary condition in equation (69), and the A_{ck} and A_{sk} values are obtained by means of the boundary equation (68). To avoid singularities in the center, we must then require that

$$m_\phi \leq \frac{k_0^2}{2} - 1 . \tag{C6}$$

Equation (C6) is a necessary condition for the magnetic tube to find a new equilibrium with the same zero-order field distributions once perturbed by the given external pressure fluctuation (C1). In the case of a potential flow, for example, $k_0 = 2$, and therefore m_ϕ must be equal to 1. An analysis similar to the one carried out in Appendix B (eqs. [B9] and [B10]) shows that equation (C6) is valid in the neighborhood of the center for general cases (i.e., also if b_0 is not a power law for all radii).

Once r_1 is known, the first-order variables Δp_1^* and $\Delta\mathscr{P}_1^*$ follow easily from the restriction of the "radial" momentum equation and its boundary condition to the vertical semiaxis $\phi = 0$: with b_1 and B_1^y defined by equations (B11) and (B12), we obtain

$$\frac{\Delta\mathscr{P}_0 \Delta\mathscr{P}_1^*}{p_{e0}} = \epsilon \frac{2\alpha'_\phi}{m'_\phi}\left[1 - \left(\frac{u}{U}\right)^{m'_\phi}\right] + \alpha_\phi P^* + \sum_{k=k_0}^{\infty} \frac{2\alpha_\phi A_{ck} k^2}{m_r + 2m_\phi}\left[1 - \left(\frac{u}{U}\right)^{m_r+2m_\phi}\right] , \tag{C7}$$

where

$$m_r = -1 - m_\phi + \sqrt{-1 + k^2 + m_\phi^2} , \tag{C8}$$

in addition to the parameters given in equation (B13).

APPENDIX D

POWER-LAW MODEL FOR A HORIZONTAL MAGNETIC FLUX TUBE SUBJECT TO GRAVITY AND AN ARBITRARY PRESSURE DISTRIBUTION ON ITS SURFACE

To zeroth order, we assume that we have the same power-law equilibrium as the one used in Appendix B, namely, equations (B2)–(B4). The only differences are that the total buoyancy is now different from zero and that the choice of the entropy excess is restricted by the fact that the total buoyancy force must be compensated by the drag force, as required by equation (80), viz.,

$$-\frac{P_{drag}}{\epsilon} = \frac{\overline{\Delta\rho_0}}{\rho_{e0}}(U) = -\frac{\Delta s_0}{c_p} - \frac{\alpha_y}{\gamma} . \tag{D1}$$

Once we have the zero-order variables, we obtain r_1 from the momentum equation. With the external stratification given by equation (B1), and the external pressure fluctuation p_{ef} of the form

$$p_{ef} = p_{e0} P_{drag} \cos \phi + \frac{b_0^2}{8\pi}(U)P(\phi) , \tag{D2}$$

where $P(\phi)$ is given in equation (C1), the solution to equation (34) and its boundary conditions is

$$r_1 = a_0(u) + a_1(u) \cos \phi + \sum_{k=k_0}^{\infty} [a_{ck}(u) \cos (k\phi) + a_{sk}(u) \sin (k\phi)] , \tag{D3}$$

with $a_0(u)$, $a_1(u)$, and $a_{ck}(u)$ and $a_{sk}(u)$ given by equations (C3), (B7), and (C4), respectively.

The other first-order variables follow from the restriction of the momentum equation and its boundary condition to the vertical semiaxis $\phi = 0$ once b_1 and B_1^y are chosen. To allow the comparison with the particular equilibria calculated in Appendices B and C, we choose b_1 and B_1^y as in Appendix B. Thus, the rest of the first-order variables describing the internal equilibrium are the same as in Appendix B, with

$$\frac{\Delta\mathscr{P}_0 \Delta\mathscr{P}_1^*}{p_{e0}} = \epsilon\left\{\frac{2\alpha'_\phi}{m'_\phi}\left[1 - \left(\frac{u}{U}\right)^{m'_\phi}\right] + C_1\left[1 - \left(\frac{u}{U}\right)\right] + C_2\left[1 - \left(\frac{u}{U}\right)^{2m_\phi + 1}\right] + C_3\left[1 - \left(\frac{u}{U}\right)^{2m_y + 1}\right]\right\} + \sum_{k=k_0}^{\infty} \frac{2\alpha_\phi A_{\mathrm{ck}} k^2}{m_r + 2m_\phi}\left[1 - \left(\frac{u}{U}\right)^{2m_\phi + m_r}\right] + P_{\mathrm{drag}} + \alpha_\phi P^* \tag{D4}$$

instead of equation (B13).

REFERENCES

Anton, V. 1983, Diplomarbeit (Germany: Univ Göttingen)
Anzer, U. 1968, Sol. Phys., 3, 298
Auslander, L., & Mckenzie, R. E. 1977, Introduction to Differentiable Manifolds (New York: Dover Publications)
Batchelor, G. K. 1967, An Introduction to Fluid Dynamics (Cambridge: Cambridge Univ. Press)
Browning, P. K., & Priest, E. R. 1983, ApJ, 266, 848
Caligari, P., Moreno-Insertis, F., & Schüssler, M. 1995, ApJ, 441, 886
Cally, P. S. 1991, J. Comput. Phys., 93, 411
Cartledge, N., & Hood, A. W. 1993, Sol. Phys., 148, 253
Defouw, R. J. 1976, ApJ, 209, 266
Dungey, J. W. 1953, MNRAS, 113, 180
Edenstrasser, J. W. 1980, J. Plasma Phys., 24, 299
Emslie, A. G., & Wilkinson, L. K. 1994, ApJ, 431, 873
Feriz-Mas, A., & Schüssler, M. 1989, Geophys. Astrophys. Fluid Dyn., 48, 217
———. 1994, ApJ, 433, 852
Giachetti, R., Van Hoven, G., & Chiuderi, C. 1977, Sol. Phys., 55, 371
Greene, J. M., Johnson, J. L., & Weimer, K. E. 1971, Phys. Fluids, 14, 671
Hu, Y. Q. 1988, ApJ, 331, 402
Hundhausen, J. R., & Low, B. C. 1994, ApJ, 429, 876
Kobayashi, S., & Nomizu, K. 1963, Foundations of Differential Geometry (New York: John Wiley & Sons)
Lerche, I., & Low, B. C. 1980, ApJ, 238, 1088
Lites, B. W., Low, B. C., Martinez Pillet, V., Seagraves, P., Skumanich, A., Frank, Z. A., Shine, R. A., & Tsuneta, S. 1995, ApJ, 446, 877
Lothian, R. M., & Hood, A. W. 1989, Sol. Phys., 122, 227
Low, B. C. 1975, ApJ, 197, 251
———. 1980, Sol. Phys., 65, 147
———. 1981, ApJ, 251, 352
———. 1993a, ApJ, 408, 689
———. 1993b, ApJ, 408, 693
Low, B. C., & Hundhausen, J. R. 1995, ApJ, 443, 818
Lüst, R., & Schlüter, A. 1954, Z. Astrophys., 34, 263
Miller, G., & Turner, L. 1981, Phys. Fluids, 24, 363
Moreno-Insertis, F. 1983, A&A, 122, 241
———. 1992, in Theory of Sunspots, ed. J. Thomas & N. Weiss (Dordrecht: Kluwer), 385
Moreno-Insertis, F., Schüssler, M., & Ferriz-Mas, A. 1992, A&A, 264, 686
Parker, E. N. 1974, ApJ, 191, 245
———. 1979, Cosmical Magnetic Fields (Oxford: Clarendon Press)
Priest, E. R. 1982, Solar Magnetohydrodynamics (Dordrecht: Reidel)
———. 1990, in Physics of Magnetic Flux Ropes, ed. C. Russel (American Geophysical Union Geophysical Monograph 58), 1
Roberts, B., & Webb, A. R. 1978, Sol. Phys., 56, 5
———. 1979, Sol. Phys., 64, 77
Tanaka, K. 1991, Sol. Phys., 136, 133
Tsinganos, K. C. 1980, ApJ, 239, 746
van Ballegooijen, A. A., & Choudhuri, A. R. 1988, ApJ, 333, 965
Webb, G. M. 1986, Sol. Phys., 106, 287
———. 1988, ApJ, 327, 933
Zweibel, E. G., & Hundhausen, A. J. 1982, Sol. Phys., 76, 261