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MHS-Equilibrium of Twisted Magnetic Tubes

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An equilibrium configuration of a horizontal twisted magnetic flux tube embedded in a stratified external medium is calculated. The equilibrium is obtained through a perturbation method which assumes a weak external stratification but which is much less restrictive than the customary thin flux tube approximation. The general equations and method of solution have been presented in an earlier paper (Emonet & Moreno-Insertis 1996). In the equilibrium obtained here, the physical variables have Gaussian-like profiles. This particular feature makes the new solution suitable as initial condition for 2D and 3D simulations of the rise of thick magnetic flux ropes through the convection zone of the Sun currently being carried out.

1. INTRODUCTION

The calculation of magnetohydrostatic (MHS) equilibrium configurations for non-force-free, twisted horizontal flux tubes is a necessary step in the study of the structure and evolution of magnetic flux tubes in the solar convection zone. The obvious application of such a calculation is to magnetic flux tubes stored in the stably stratified regions of the solar interior. Perhaps more importantly, such study can be applied also to magnetic rings rising in a quasi-static manner through the convection zone, (e.g., with the drag force continuously compensating the total buoyancy force of the tube). The latter application is of particular interest as a prerequisite for numerical calculations of the rise of magnetic flux tubes in the uppermost scale-heights of the convection zone, where departures from the thin flux tube approximation become important.

In a recent paper, Emonet & Moreno-Insertis (1996, hereafter paper I) have presented solutions for the MHS structure of non-force-free twisted horizontal magnetic flux tubes including gravity and an arbitrary pressure perturbation on the tube boundary. To simplify the problem, it was assumed that the physical quantities along the tube axis were invariant. The resulting 2D free-boundary problem was then solved by using general non-orthogonal flux coordinates and by considering the case of weak stratification (pressure scale-height larger than the tube radius). This permits introduction of a perturbation scheme which is much less restrictive than the customary slender flux tube approximation: for instance, it does not impose any limitation on the strength of the azimuthal field as compared to the longitudinal field. Using the equations and techniques of paper I, one can study, in particular, the mutual dependence of: (1) the (differential) buoyancy in the tube, (2) the azimuthal field (intensity and field line geometry), (3) the gas pressure and longitudinal field distributions. In that paper it is shown how the differences in buoyancy in the tube cross section force the closed azimuthal field lines in the interior of the tube to shift vertically while the gas pressure stays hydrostatically stratified along the azimuthal field lines. The effect of a flow around the tube is found to be twofold: external pressure fluctuations with circular wavenumber greater than 1 force the field lines to bend to achieve the equilibrium, whereas the cosine and sine components produce non-vanishing resulting forces on the tube ('drag' and 'lift', respectively).

In paper I, application of the equations to the calculation of power-law profiles of the physical quantities in the tube interior was made. Power laws are of interest, among other things, because they provide a natural generalization of the radial expansion of the physical quantities. They also permit the study of shell-like structures that may develop along the tube evolution. For the study of the rise of thick flux tubes, however, in particular for the corresponding numerical calculations, it is equally important to have in hand equilibrium configurations in which the physical quantities have Gaussian shapes. A flux tube with this kind of pressure profiles is of particular interest as an initial condition in numerical simulations, especially if the numerical scheme used to integrate the MHD equations is of Lax-Wendroff type (see, e.g., Morton & Mayers, 1994, Section 4.5).

In the present paper we use the equations of paper I to obtain an equilibrium with Gaussian shapes for the gas- and longitudinal magnetic field pressure profiles. The contents is as follows: in Sect. 2 we briefly lay out the equations and method of solution. In Sect. 3 the Gaussian solution is obtained. The conclusions are summarized in Sect. 4.

2. MHS EQUATIONS

We want to calculate the static equilibrium of a horizontal magnetic flux tube embedded in a stratified external medium. Instead of using an orthogonal system of coordinates, we introduce the non-orthogonal set (u, ϕ, y) (see Fig. 1) in which u is constant along the lines of force of the transverse field $\mathbf{B}_t$ $(= \mathbf{B} - B^y \mathbf{e}_y)$. The value of u is fixed by requiring that $u = r = z$ along the vertical semiaxis $\phi = 0$.

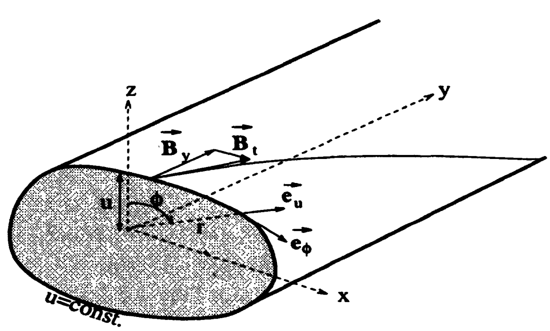

FIGURE 1. The coordinates and some of the symbols used in the paper

The set (u,ϕ,y) has the advantage of strongly simplifying the boundary condition: the surface of the tube corresponds to one of the $u = const$ surfaces, $u = U$ say. The transformation of the solenoidality equation and of the equilibrium equation from the Cartesian coordinates to (u,ϕ,y) is detailed in paper I. Here we just write the results.

In the new unit basis $(\mathbf{e}_u, \mathbf{e}_\phi, \mathbf{e}_y)$, $\mathbf{B}_t$ has a single component, $\mathbf{B}_t = B^\phi \mathbf{e}_\phi$. This permits immediate integration of the solenoidality equation. In fact, the latter becomes identical to the requirement that there exists a function $b(u)$ such that

$$B^\phi = \frac{\sqrt{r_{,\phi}{}^2 + r^2}}{r\, r_{,u}}\, b(u)\,, \tag{1}$$

where b has the dimension of a magnetic field. The subscripts $,u$ and $,\phi$ indicate the partial derivatives with respect to u and ϕ. Next, the scalar product of the momentum equation with $\mathbf{e}_y$ implies that B^y be a function of u only, $B^y = B^y(u)$, whereas the scalar product with $\mathbf{e}_u$ and $\mathbf{e}_\phi$ give after some simplifications:

$$0 = -\frac{\partial}{\partial u}\left(\Delta p + \frac{B^{y\,2}}{8\pi} + \frac{b^2}{8\pi}F\right) - \Delta\rho\, G\frac{\partial z}{\partial u} + \frac{b^2}{4\pi}Q\,, \tag{2}$$

$$0 = -\frac{\partial \Delta p}{\partial \phi} - \Delta\rho\, G\frac{\partial z}{\partial \phi}\,, \tag{3}$$

respectively. The functionals F and Q are:

$$F = \frac{r_{,\phi}{}^2 + r^2}{r^2 r_{,u}{}^2}\,, \quad Q = \frac{\partial F}{\partial \phi}\,\frac{r_{,u} r_{,\phi}}{2\left(r_{,\phi}{}^2 + r^2\right)} - \frac{\sqrt{F}}{R_c}\,, \quad R_c = \frac{\left(r_{,\phi}{}^2 + r^2\right)^{3/2}}{\left(2r_{,\phi}{}^2 - r r_{,\phi\phi} + r^2\right)}\,. \tag{4}$$

G is the modulus of the gravitational acceleration $\mathbf{G}$, R_c is the radius of curvature of the $\mathbf{B}_t$ line of force, and the coordinate z is an unknown function of u and ϕ, viz. $z = r(u,\phi)\cos\phi$. In these equations we have used the definitions

$$\Delta p(u,\phi) = p(u,\phi) - p_e\left[z\,(u,\phi)\right]\,, \quad \Delta\rho(u,\phi) = \rho(u,\phi) - \rho_e\left[z\,(u,\phi)\right]\,, \tag{5}$$

instead of p and ρ and subtracted the external stratification from the internal equilibrium equations. Eq. (3) is equivalent to the condition of hydrostatic stratification of the gas along the transverse magnetic field lines. Equations (1) through (4) are complemented with the two boundary conditions:

$$0 = \Delta p + \frac{B^{y\,2}}{8\pi} + \frac{b^2}{8\pi}F\,, \quad \text{at } u = U \tag{6}$$

$$u = r\,, \quad \text{for } \phi = 0\,, \tag{7}$$

and the equation of state. Finally, for equilibrium in the absence of external flows, the total buoyancy of the tube has to be zero:

$$0 = \mathbf{G}\int_0^{2\pi}\int_0^U \Delta\rho\, r r_{,u}\, du\, d\phi\,. \tag{8}$$

To solve the nonlinear system of equations (1)–(8) we assume that our tube has a small radius, U, as measured by the local pressure scaleheight H_e, i.e. $U/H_e \ll 1$. In this limit, the gravitational terms in the equations are only a perturbation to the equilibrium. Thus, we can consider, first, an axisymmetric magnetic flux tube in equilibrium with a background plasma of constant pressure ($\mathbf{G} = 0$), and second, the same equilibrium plus a linear correction due to gravity. To that end, we define for each physical quantity a zero- and a first-order symbol:

$$\Delta p = \Delta p_0(u)\ [1 + \Delta p_1(u,\phi)]\ ,\ p_e = p_{e0}\ [1 + p_{e1}(u,\phi)]\ ,\ r = u\ [1 + r_1(u,\phi)]\ ,\ldots \tag{9}$$

and introduce them in the equations (1) through (8). To zero order ($\mathbf{G} = 0$) we obtain the customary equations describing the equilibrium of an axisymmetric flux tube embedded in an external medium of constant pressure p_{e0} (see paper I and Parker 1979). To first order, Eq. (1) transforms into:

$$B_1^\phi = b_1 - \frac{\partial}{\partial u}(u r_1)\ , \tag{10}$$

whereas the momentum equations (2)–(3) together with their boundary conditions (6)–(7) lead to:

$$0 = -\frac{\partial \Delta \rho_0}{\partial u} G u \sin\phi + \frac{\partial}{\partial u}\left(\frac{{b_0}^2}{4\pi}\frac{\partial\,(u r_{1,\phi})}{\partial u}\right) + \frac{{b_0}^2}{4\pi}\frac{1}{u}\left(\frac{\partial\,(u r_{1,\phi})}{\partial u} + r_{1,\phi} + r_{1,\phi\phi\phi}\right) \tag{11}$$

$$0 = -\Delta p_0 \frac{\partial \Delta p_1}{\partial \phi} + \Delta \rho_0\, G\, u\, \sin\phi \tag{12}$$

$$0 = \Delta p_0 \frac{\partial \Delta p_1}{\partial \phi} - \frac{{b_0}^2}{4\pi}\frac{\partial\,(u r_{1,\phi})}{\partial u}\ , \quad \text{at } u = U \tag{13}$$

$$0 = r_1\ , \quad \text{for } \phi = 0\ . \tag{14}$$

(11) and (13) do not result from the direct linearization of the 'radial' momentum equation (2) and of its corresponding boundary condition (6). In fact, they are the first order of the ϕ-derivative of (2) and (6), respectively. Thus, they must be completed with the restriction of (2) and (6) to the vertical semi-axis $\phi = 0$. Finally, the integral condition (8) yields:

$$0 = G \int_0^U \Delta \rho_0\, u\, du\ . \tag{15}$$

3. THE GAUSSIAN MODEL

To zero order we have the customary radial momentum equation together with its boundary condition for the three unknowns $\Delta p_0(u)$, $b_0(u) = B_0^\phi(u)$ and $B_0^y(u)$ ($\Delta \rho_0(u)$ is obtained later from thermodynamics considerations). We have therefore two functional degrees of freedom; we set b_0 and B_0^y, and calculate Δp_0:

$$\frac{{B_0^\phi}^2}{8\pi p_{e0}} = \frac{b_0^2}{8\pi p_{e0}} = m\,\frac{1}{1-c}\,\frac{u^2}{w^2}\,e^{-u^2/w^2}\ , \tag{16}$$

$$\frac{{B_0^y}^2}{8\pi p_{e0}} = n\,\frac{1}{1-c}\left(e^{-u^2/w^2} - \chi c\right)\ , \tag{17}$$

$$\frac{\Delta p_0}{p_{e0}} = \frac{1}{1-c}\left(\left(m - n - m\frac{u^2}{w^2}\right)e^{-u^2/w^2} - (m - n\chi)\,c\right)\ . \tag{18}$$

$c = \exp(-U^2/w^2)$, and w, m, n and χ are adjustable parameters. It can easily be verified that the resulting total pressure excess (the sum of (16) through (18)) has a Gaussian shape (see Fig. 2). Next, $\Delta\rho_0$ is related to Δp_0 by assuming that the entropy is constant in the tube ($s_0 = const$ and $s_1 = 0$). One could solve the problem for a more general thermodynamic state of the tube. However, this one has the advantage of being simple and reasonable for tubes which rise through the convection zone of the Sun (Moreno-Insertis 1983, paper I). Thus,

$$\frac{\Delta\rho_0}{\rho_{e0}} = e^{-\Delta s_0/c_p}\left(1 + \frac{\Delta p_0}{p_{e0}}\right)^{1/\gamma} - 1 \approx -\frac{\Delta s_0}{c_p} + \frac{1}{\gamma}\frac{\Delta p_0}{p_{e0}}\,, \tag{19}$$

with Δs_0 fixed by the no-buoyancy condition (15). The approximate expression on the right is only valid for large enough plasma beta and small enough entropy difference between inside and outside the tube (which are good approximations for tubes in the convection zone except its uppermost levels; Moreno-Insertis 1983, paper I).

Once the zero-order equilibrium is known, the first-order system of equations (11)–(14) provides two equations with two boundary conditions for the five unknowns $\Delta p_1(u,\phi)$, $\Delta\rho_1(u,\phi)$, $b_1(u)$, $B_1^y(u)$ and $r_1(u,\phi)$. Thus, with the assumption of constant entropy which gives $\Delta\rho_1$ as function of Δp_1 there remain two functional degrees of freedom. For simplicity we choose $B_1^y = b_1 = 0$. As shown in paper I, without restriction of the generality we can look for a solution r_1 to Eq. (11) of the form

$$r_1(u,\phi) = a(u)\,(\cos\phi - 1)\ . \tag{20}$$

The introduction of (20) in (11) and two subsequent integrations with respect to u yield:

$$\begin{aligned} a(u) &= -\frac{1}{u}\int_0^u \frac{4\pi}{b_0^2}\frac{G}{u'}\left[\int_0^{u'} \frac{\partial\Delta\rho_0}{\partial u''}u''^2\,du''\right]du' \\ &= -\frac{1}{4\gamma}\frac{w}{H_{e0}}\frac{n}{m}\left[\left(\frac{1}{2} - \frac{m}{n}\right)\frac{u}{w} + A\left(\frac{u}{w}\right)\right]\,, \end{aligned} \tag{21}$$

where $A(t) = \dfrac{1 + t^2 - t^4/2 - e^{t^2}}{t^3} + \dfrac{\mathrm{Ei}\,(t^2) - \ln\,(t^2) - C_e}{t} = \displaystyle\sum_{i=3}^{\infty}\frac{t^{2i-3}}{(i-1)\,i!}\,,$

$C_e \cong 0.5772$ is Euler's constant, and $\mathrm{Ei}(t) = \int_{-\infty}^{t} e^v/v\,dv$ (see Abramowitz & Stegun 1964). Once $r_1(u,\phi)$ is known, B_1^ϕ is given by (10). The integration with respect to ϕ of the 'azimuthal' momentum equation (12) is immediate:

$$\Delta\rho_0\Delta p_1 = \Delta\rho_0\Delta p_1^* + \Delta\rho_0 G u\,(1 - \cos\phi)\ . \tag{22}$$

$\Delta p_1^*(u) \overset{\text{def}}{=} \Delta p_1(u, \phi = 0)$ is obtained from the restriction of the 'radial' momentum equation and its boundary condition to the vertical semi-axis $\phi = 0$:

$$\begin{aligned} \frac{\Delta p_0\Delta p_1^*}{p_{e0}} &= \frac{1}{p_{e0}}\int_u^U \left[\Delta\rho_0 G + \frac{b_0^2}{4\pi}\frac{a\,(u)}{u}\right]du = \frac{1}{\gamma\,(1-c)}\frac{w}{H_{e0}}\Bigg\{ \\ &\left(\gamma\,(1-c)\frac{\Delta s_0}{c_p} + c\,(m - \chi n)\right)\frac{u - U}{w} + \frac{9n - 6m}{16}\sqrt{\pi}\left[\mathrm{erf}\left(\frac{u}{w}\right) - \mathrm{erf}\left(\frac{U}{w}\right)\right] \\ &- \left(\frac{n}{8} + \frac{m}{4}\right)\left[\frac{u}{w}e^{-u^2/w^2} - \frac{U}{w}e^{-U^2/w^2}\right] - \frac{n}{2}\int_{u/w}^{U/w} t\,A(t)\,e^{-t^2}dt \Bigg\}\,. \end{aligned} \tag{23}$$

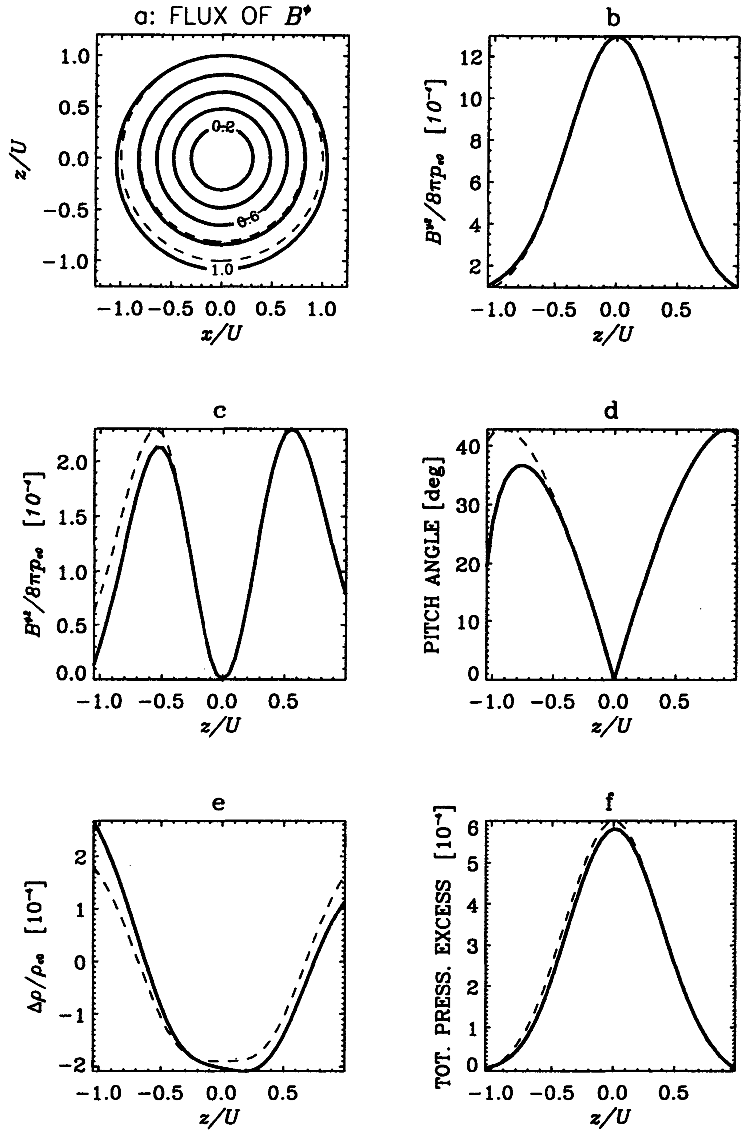

FIGURE 2. Horizontal magnetic flux tube in static equilibrium with a stratified polytropic atmosphere. The first plot represents the isolines of the transverse magnetic flux normalized to its boundary value. The other plots are the profiles of (b) ${B^y}^2/8\pi p_{e0}$, (c) ${B^\phi}^2/8\pi p_{e0}$, (d) $\arctan(B^\phi/B^y)$, (e) $\Delta\rho/\rho_{e0}$ and (f) $\Delta p/p_{e0} + {B^y}^2/8\pi p_{e0} + {B^\phi}^2/8\pi p_{e0}$ along the vertical z-axis (full lines). The zero-order profiles are over-plotted (dashed lines). The core of the tube is lighter than the outer layers. The resulting differential buoyancy is counteracted by the reorganization of the 'azimuthal' field.

The density excess can then be calculated using the condition of constant entropy:

$$\frac{\Delta\rho_0\Delta\rho_1}{\rho_{e0}} = \frac{1}{\gamma}\frac{\Delta p_0\Delta p_1}{p_{e0}} + \left(\frac{\Delta\rho_0}{\rho_{e0}}\rho_e - \frac{1}{\gamma}\frac{\Delta p_0}{p_{e0}}p_{e1}\right) + \frac{s_{e0}}{c_p}s_{e1}\ , \tag{24}$$

with $p_{e1}(u,\phi)$, $\rho_{e1}(u,\phi)$ and $s_{e1}(u,\phi)$ given by the external stratification.

4. CONCLUSION

We have calculated the MHS equilibrium of a twisted horizontal magnetic flux rope in which the physical variables present Gaussian-like profiles. For brevity we have not included the effects of an external flow, but these can easily be calculated following the general equations of paper I.

The present solution has been used as initial condition in the 2D numerical simulation of the rise of thick magnetic tubes across the uppermost 20,000 km of the convection zone. In this region the departure of the inner structure of the tube from the thin flux tube approximation must become apparent. This will be increasingly the case as the tube approaches the photosphere. Gaussian initial conditions have favorable properties as regards the development of numerical instabilities in the subsequent evolution of the magnetic tube: this is known from general theoretical considerations and has been shown also in the actual numerical calculations. The results of these calculations will be published in a paper in preparation.

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